Symmetric Powers of Modular Representations for Groups with a Sylow Subgroup of Prime Order

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Abstract

Let V be a representation of a finite group G over a field of characteristic p. If p does not divide the group order, then Molien's formula gives the Hilbert series of the invariant ring. In this paper we find a replacement of Molien's formula which works in the case that $|G|$ is divisible by p but not by p^2. We also obtain formulas which give generating functions encoding the decompositions of all symmetric powers of V into indecomposables. Our methods can be applied to determine the depth of the invariant ring without computing any invariants. This leads to a proof of a conjecture of the second author on certain invariants of $\mathrm{GL}_2(p)$.

Contents

Introduction

One of the most remarkable tools in the invariant theory of finite groups is Molien's formula, which allows the computation of the dimensions of all homogeneous invariant subspaces without touching a single invariant. More precisely, if $K[V]^G$ is the invariant ring of a finite group G acting on a finite-dimensional vector space V over a field K of characteristic zero, then the Hilbert series $H\left(K[V]^G,t\right) := \sum_{d=0}^{\infty} \dim(K[V]_d^G) \cdot t^d$ is given by

$$H\left(K[V]^G,t\right) = \frac{1}{|G|} \sum_{\sigma\in G} \frac{1}{\det_V(1-\sigma t)}. \tag{0.1}$$

There is a straightforward generalization to the case that K has positive characteristic p not dividing the group order. However, Molien's formula fails in the modular case, where p divides $|G|$.

Usually the proof of Molien's formula proceeds in two steps: First one shows that the coefficient of t^d in $1/\det_V(1-\sigma t)$ is the trace of the action of σ on the d-th symmetric power $S^d(V) = K[V]_d^G$ (the space of homogeneous polynomials of degree d in $K[V]$). Then one uses the fact that for a representation U with character χ_U the dimension of the invariants is given by

$$\dim(U^G) = \frac{1}{|G|} \sum_{\sigma \in G} \chi_U(\sigma). \tag{0.2}$$

In modular representation theory we can use Brauer characters instead of ordinary characters. Let us call the map which assigns to a modular representation its Brauer character evaluated at a fixed p-regular element $\sigma \in G$ a p-regular species. It is easy to evaluate these species at symmetric powers $S^d(V)$ of V. However, p-regular species do not contain enough information to yield the dimension of the invariant subspace. (They can be used to calculate a generating function encoding the multiplicities of a simple module as a composition factor in the symmetric powers, see Mitchell [13, Proposition 1.2].) Therefore we need to consider other species (ring homomorphisms from the representation ring to $\mathbb{C}$), which we call p-singular. The evaluation of these species at symmetric powers $S^d(V)$ is quite difficult. In this paper we solve this problem in the case that $|G|$ is divisible by $p = \operatorname{char}(K)$ but not by p^2. This yields expressions which bear some similarity to the quotients $1/\det_V(1-\sigma t)$ in Molien's formula (see Theorem 1.14 on page 9). As in the proof of Molien's formula, we then need linear combinations of the species which yield the dimension of the invariant subspace and thus give an analogue to (0.2). We do more than this by giving averaging operators which assign to a representation U the multiplicity of some (fixed) indecomposable representation as a direct summand of U. Combining these results, we obtain formulas for the Hilbert series of the invariant ring (see Theorem 2.4 on page 16), for generating functions encoding the multiplicities of indecomposables as direct summands of the symmetric powers, and for the Hilbert series of the cohomology modules with coefficients in $K[V]$. Our formulas are somewhat lengthy to write down, but in terms of computational difficulty they are almost as easy to evaluate as Molien's formula. In particular, no computation of any invariants is necessary to find the Hilbert series. Our formulas were implemented in MAGMA (see Bosma et al. [4]) by Denis Vogel.

This work was very much inspired by the ground-breaking paper of Almkvist and Fossum [1], who obtained complete information about the decomposition of the exterior and symmetric powers of an indecomposable representation of the cyclic group P of order p. In [8] we presented their arguments in a much shorter way and extended their results to arbitrary representations of $P \times H$, where H is a finite group of order coprime to p. We will frequently refer to [8] in this paper.

In the first section we assume that G is a finite group with a normal Sylow p-subgroup of order p. We determine the representation ring of KG and from this the species. Using the theory of lambda-rings and a periodicity property, we find formulas for the evaluation of the species at the sigma-series (which encode the symmetric powers). Then we construct $\mathbb{C}$-linear combinations of the species which give the multiplicity of an indecomposable summand and the dimension of the invariant subspace. In the second section we drop the assumption that the Sylow p-subgroup be normal. We use a special case of the Green correspondence to transport results from the first section to the more general case. As a corollary we prove that the degree of the Hilbert series is bounded from above by $-\dim(V)$. We take an elaborate look at the example $G = \mathrm{SL}_2(p)$ and describe a method for deriving formulas which for a fixed n give the Hilbert series of the invariants of $\mathrm{SL}_2(p)$ acting on binary forms of degree n, or, more generally, on a semisimple KG-module. These formulas are for general p. We give some examples (see Example 2.7 on page 19). Finally, we apply our methods to the calculation of the Hilbert series of the cohomology modules $H^i(G, K[V])$. Using results from Kemper [11], we obtain an easy method to determine the depth of the invariant ring for G with a Sylow p-subgroup of order p. As an application, we prove a conjecture of the second author which states that the invariant ring of $\mathrm{GL}_2(p)$ acting on binary forms of degree at most 3 is always Cohen-Macaulay.

1 Groups with a normal Sylow p-subgroup of order p

In this section G denotes a finite group with a normal Sylow p-subgroup $P = \langle \pi \rangle$ of order p, and K a splitting field of G with $\mathrm{char}(K) = p$. By the theorem of Schur-Zassenhaus $G \cong P \rtimes H$ with $H \leq G$ of order coprime to p. We fix a complement H of P in G. In this article KG denotes the group algebra, and all tensor products are over K. We will develop methods to find the Hilbert series of the invariant ring for a representation of G over K, along with the generating functions encoding the multiplicities of an indecomposable module in the symmetric powers. In Section 2 we will drop the assumption that P is normal.

1.1 The representation ring and species

We first determine the representation ring of KG and then find the species. For this purpose it is convenient to label the simple KH-modules by their Brauer characters. More precisely, we choose and fix a p-modular system (K_0, R, K) with $K_0 \subset \mathbb{C}$ (see Curtis and Reiner [5, p. 402]). Assigning to each simple KH-module its Brauer character (with respect to (K_0, R, K); see Curtis and Reiner [5, Definition 17.4]) yields a bijection between the simple KH-modules and the set $\mathrm{Irr}(H)$ of irreducible ordinary characters of H. We write V_χ for the simple KH-module with Brauer character χ. V_χ becomes a KG-module via $G \to G/P \cong H$. A Brauer character which is of special interest is given as follows: For $\tau \in H$ we have

$$\tau\pi\tau^{-1} = \pi^{l_\tau}$$

with $l_\tau \in \mathbb{Z}$. Let $^-$ denote the natural map from $\mathbb{Z}$ to K. Then a one-dimensional KH-module is given by $\tau \mapsto \overline{l_\tau} \in \mathrm{GL}_1(K)$. We denote the Brauer character of this module by $\alpha \in \mathrm{Irr}(H)$.

Further KG-modules are obtained by forming the quotients

$$V_n := KP\,/KP(\pi - 1)^n \quad (1 \leq n \leq p).$$

These are KP-modules, which become KG-modules by

$$\tau\,(w + KP(\pi - 1)^n) := \tau w \tau^{-1} + KP(\pi - 1)^n \quad \text{for} \quad \tau \in H.$$

We can now list the indecomposable KG-modules and their composition series.

Proposition 1.1. *The KG-modules*

$$V_{n,\chi} := V_n \otimes V_\chi \quad (1 \leq n \leq p,\ \chi \in \mathrm{Irr}(H))$$

form a complete set of pairwise non-isomorphic indecomposable KG-modules. $V_{n,\chi}$ is projective if and only if $n = p$. Moreover, $V_{n,\chi}$ is uniserial with composition factors (from top to bottom)

$$V_\chi, V_{\alpha\chi}, \ldots, V_{\alpha^{n-1}\chi}.$$

Proof. We first remark that for any non-zero KG-module V the P-invariants V^P form a non-zero submodule. Assume that $V_{n,\chi} = U_1 \oplus U_2$ with U_i non-zero. Then

$$V_\chi \cong V_{n,\chi}^P = U_1^P + U_2^P,$$

in contradiction with the simplicity of V_χ. Hence the $V_{n,\chi}$ are indecomposable. In Alperin [2, p. 42] we find that all indecomposables are uniserial, and there are ep non-isomorphic indecomposables, where e is the number of simple KG-modules. Now for any module V the P-invariants V^P form a non-zero submodule, hence $S = S^P$ for a simple module S, so S is in fact a KH-module.

Therefore the number e equals the number $|\operatorname{Irr}(H)|$ of simple KH-modules, and we conclude that all indecomposables are given by the $V_{n,\chi}$. The fact that $V_{n,\chi}$ is projective if and only if $n = p$ follows from Alperin [2, Corollary 3, p. 66].

The composition factors of $V_{n,\chi}$ are $\left((\pi-1)^i KP \otimes V_\chi\right) / \left((\pi-1)^{i+1} KP \otimes V_\chi\right)$ (for $0 \le i < n$), and it is easily verified that for $\tau \in H$ we have

$$\tau\left((\pi-1)^i + (\pi-1)^{i+1}KP\right) = \overline{l_\tau}^{\,i}(\pi-1)^i + (\pi-1)^{i+1}KP.$$

□

In order to describe the representation ring we need to acquire some knowledge on tensor products.

Lemma 1.2. *We have*

$$V_2 \otimes V_n \cong \begin{cases} V_{n-1,\alpha} \oplus V_{n+1} & \text{if} \quad 0 < n < p, \\ V_{p,\alpha} \oplus V_p & \text{if} \quad n = p, \end{cases}$$

where we set $V_0 := 0$.

Proof. This follows from Feit [7, Chapter VIII]. More precisely, we use Theorem 2.7(i) for $n < p-1$, Theorem 2.7(iii) for $n = p-1$, and Lemma 2.2 for $n = p$. □

We denote the representation ring (or Green ring) of a group S over a field K by R_{KS}. This is the free $\mathbb{Z}$-module generated by the isomorphism classes of indecomposable KS-modules, with multiplication given by the tensor product (see [8, Definition 1.8]). R_{KH} is naturally embedded into R_{KG}, and Proposition 1.1 and Lemma 1.2 tell us that R_{KG} is generated by V_2 as an algebra over R_{KH}. The next goal is to obtain a presentation of R_{KG}. For this purpose (and even more for things to follow in later sections) it is very convenient to assume that α has a square root $\beta \in \operatorname{Irr}(H)$. If not, we substitute G by a group extension $\widetilde{G}$ such that a homomorphism $\beta\colon \widetilde{G} \to \mathbb{C}^*$ exists, making the diagram

$$\begin{tikzcd} \widetilde{G} \arrow[r, "\beta"] \arrow[d] & \mathbb{C}^* \arrow[d, "\square"] \\ G \arrow[r, "\alpha"] & \mathbb{C}^* \end{tikzcd}$$

commutative. Here $\square$ is the map sending x to x^2. $\widetilde{G}$ can be chosen as the pullback from the diagram, i.e.,

$$\widetilde{G} = \{(\sigma, x) \in G \times \mathbb{C}^* \mid \alpha(\sigma) = x^2\}.$$

If α already has a square root in $\operatorname{Irr}(H)$, we set $\widetilde{G} := G$. For example, in the case $p = 2$ we have $\alpha = 1$, so $\widetilde{G} = G$. Thus we can choose $\widetilde{G}$ to have a normal Sylow p-subgroup of order p. We now substitute G by $\widetilde{G}$, i.e., **we assume for the rest of this section that $\beta \in \operatorname{Irr}(H)$ exists with $\beta^2 = \alpha$.** It is furthermore convenient to add an element μ with $V_2 = \beta(\mu + \mu^{-1})$ to R_{KG}. More precisely, we form the polynomial ring $R_{KG}[T]$ and then the quotient ring

$$R_{KG}[\mu] := R_{KG}[T]/(T^2 - \beta^{-1}V_2T + 1).$$

Here by abuse of notation we write χ for V_χ if χ is a one-dimensional character.

Lemma 1.3. *In $R_{KG}[\mu]$ we have*

$$V_n = \beta^{n-1}\frac{\mu^n - \mu^{-n}}{\mu - \mu^{-1}} \quad (1 \le n \le p),$$

where the fraction is short for $\sum_{j=0}^{n-1} \mu^{n-1-2j}$.

Proof. The proof is analogous to the one of Lemma 2.3 in [8]. In fact, the result is true for $n = 1$ and $n = 2$. For $1 < n < p$ we have by Lemma 1.2 and by induction

$$V_{n+1} = V_2 V_n - \alpha V_{n-1} = \beta(\mu + \mu^{-1}) \cdot \beta^{n-1} \frac{\mu^n - \mu^{-n}}{\mu - \mu^{-1}} - \beta^n \frac{\mu^{n-1} - \mu^{-(n-1)}}{\mu - \mu^{-1}} = \beta^n \frac{\mu^{n+1} - \mu^{-(n+1)}}{\mu - \mu^{-1}},$$

which proves the lemma. □

Theorem 1.4. *Let $R_{KG}[\mu]$ be defined as above. Then the map of R_{KH}-algebras given by*

$$R_{KH}[X] \to R_{KG}[\mu], \;\; X \mapsto \mu$$

is surjective, and the kernel is the principal ideal generated by

$$F(X) := (X - \beta)(X - \beta^{-1}) \frac{X^{2p} - 1}{X^2 - 1}.$$

Proof. This is analogous to Theorem 2.4 in [8]. Using Lemma 1.3 we see that $F(X)$ maps to

$$(\mu - \beta)(\mu - \beta^{-1}) \frac{\mu^{2p} - 1}{\mu^2 - 1} = \mu^p \beta^{-p} (V_2 - 1 - \alpha) V_p,$$

which is zero by Lemma 1.2. Now we see that μ^{-1} lies in the image of the map, hence it is surjective. The result follows by comparing the $\mathbb{Z}$-ranks of $R_{KH}[X]/(F)$ and $R_{KG}[\mu]$. □

A species of a group S over a field K is a non-zero ring-homomorphism from the representation ring R_{KS} to $\mathbb{C}$.

Remark 1.5. If $\mathrm{char}(K) = p$ and the Sylow p-subgroups of a group S are cyclic, then by a theorem of J. A. Green and M. F. O'Reilly the complexification $\widetilde{R}_{KS} := \mathbb{C} \otimes_{\mathbb{Z}} R_{KS}$ of the representation ring has no nilpotent elements (see Benson [3, Theorem 5.8.7]). $\widetilde{R}_{KS}$ is an m-dimensional $\mathbb{C}$-algebra, where m is the number of indecomposable KS-modules. Therefore $\widetilde{R}_{KS}$ is the coordinate ring of a variety consisting of m points. The points correspond to the ring-homomorphisms $\widetilde{R}_{KS} \to \mathbb{C}$. Restriction to R_{KS} and tensoring with $\mathbb{C}$ provide a bijective correspondence between ring-homomorphisms $\widetilde{R}_{KS} \to \mathbb{C}$ and species of S over K. Thus we conclude that there are exactly m species of S over K.

The species of S over K are linearly independent over $\mathbb{C}$. This follows (for example) from the interpretation of $\widetilde{R}_{KS}$ as the coordinate ring of a zero-dimensional variety. ◁

We finish this section by giving the species of G over K. We write $M_n \subset \mathbb{C}^*$ for the set of n-th roots of unity.

Theorem 1.6. *For $\tau \in H$ and $\gamma \in (M_{2p} \setminus \{\pm 1\}) \cup \{\beta(\tau), \beta(\tau)^{-1}\}$ there exists a unique ring-homomorphism $\phi_{\tau,\gamma} \colon R_{KG}[\mu] \to \mathbb{C}$ with*

$$\phi_{\tau,\gamma}(V_\chi) = \chi(\tau) \quad \textit{for} \quad \chi \in \mathrm{Irr}(H), \quad \textit{and} \quad \phi_{\tau,\gamma}(\mu) = \gamma.$$

The restriction of $\phi_{\tau,\gamma}$ to R_{KG} is a species of G over K, and $\phi_{\tau,\gamma}$ and $\phi_{\tau',\gamma'}$ restrict to the same species if and only if τ and τ' are conjugate in H and $\gamma' = \gamma^{\pm 1}$. All species of G over K are given by homomorphisms of the form $\phi_{\tau,\gamma}$.

Proof. Sending V_χ to $\chi(\tau)$ gives a ring-homomorphism $R_{KH} \to \mathbb{C}$. This can be extended to a unique ring-homomorphism $R_{KH}[X] \to \mathbb{C}$ by sending X to γ. By the choice of γ, the relation $F(X)$ from Theorem 1.4 is mapped to 0. Hence by Theorem 1.4, $\phi_{\tau,\gamma}(V_\chi)$ exists and is unique.

If τ and τ' are conjugate in H, then $\chi(\tau) = \chi(\tau')$ for all $\chi \in \mathrm{Irr}(H)$, and if moreover $\gamma' = \gamma^{\pm 1}$ then $\phi_{\tau,\gamma}(V_n) = \phi_{\tau',\gamma'}(V_n)$ by Lemma 1.3. Hence $\phi_{\tau,\gamma}$ and $\phi_{\tau',\gamma'}$ restrict to the same species of

G. Conversely, if $\phi_{\tau,\gamma}$ and $\phi_{\tau',\gamma'}$ restrict to the same species, then τ and τ' are conjugate in H by character theory. Moreover, evaluation at V_2 yields $\gamma+\gamma^{-1}=\gamma'+(\gamma')^{-1}$, hence the polynomials $(X-\gamma)(X-\gamma^{-1})$ and $(X-\gamma')(X-(\gamma')^{-1})$ are equal, and we have $\gamma'=\gamma^{\pm 1}$. Counting the species given by $\phi_{\tau,\gamma}$ and comparing with the number of indecomposable KG-modules yields that the $\phi_{\tau,\gamma}$ are all species of G over K. □

Remark 1.7. The species found in Theorem 1.6 naturally subdivide into two classes: those with $\gamma\in\{\beta(\tau)^{\pm 1}\}$ and those with $\gamma\in M_{2p}\setminus\{\pm 1\}$. Consider a species $\phi_{\tau,\beta(\tau)^{\pm 1}}$ from the first class. By Lemma 1.3 we have

$$\phi_{\tau,\beta(\tau)^{\pm 1}}(V_{n,\chi})=\chi(\tau)\beta(\tau)^{n-1}\sum_{j=0}^{n-1}\beta(\tau)^{\pm(n-1-2j)}=\chi(\tau)\sum_{j=0}^{n-1}\alpha(\tau)^j.$$

Comparing this to the composition series of $V_{n,\chi}$ given in Proposition 1.1, we recognize $\phi_{\tau,\beta(\tau)^{\pm 1}}(V_{n,\chi})$ as the Brauer character of $V_{n,\chi}$ evaluated at τ. Since every p-regular conjugacy class of G has a representative in H, we conclude that the species in the first class are exactly the species arising from p-regular classes by evaluation of Brauer characters. These "**p-regular**" species are complemented by the "**p-singular**" species $\phi_{\tau,\gamma}$ with $\gamma\in M_{2p}\setminus\{\pm 1\}$, whose origin is less obvious. ◁

1.2 Exterior and symmetric powers

In this section we study exterior and symmetric powers of a KG-module, where G is as introduced at the beginning of Section 1. It is convenient to encode these into power series over the representation ring R_{KG}.

Definition 1.8. *For S a finite group, K a field and V a finite-dimensional KG-module we define*

$$\sigma_t(V):=\sum_{r=0}^{\infty}S^r(V)\cdot t^r \quad \text{and} \quad \lambda_t(V):=\sum_{r=0}^{\infty}\Lambda^r(V)\cdot t^r$$

as elements of the formal power series ring $R_{KS}[[t]]$. Here $S^r(V)$ is the r-th symmetric power of V, and $\Lambda^r(V)$ is the r-th exterior power.

It is quite hard to determine symmetric and exterior powers. However, we do have some rules, which are derived from the theory of symmetrization. This theory provides a machinery to translate an identity of symmetric polynomials into an equivalence of symmetrization functors (see, for example, [8, Section 1]). In the setting of Definition 1.8, the first rule we will use is

$$\sigma_t(V)\cdot\lambda_{-t}(V)\equiv 1 \mod t^p \tag{1.1}$$

if $p:=\operatorname{char}(K)>0$ (see [8, Theorem 1.11]; if the characteristic of K does not divide $|S|$, the above congruence becomes an equality). In order to formulate a rule which relates $\lambda_t(U\otimes V)$ to $\lambda_t(U)$ and $\lambda_t(V)$, we need a further definition.

Definition 1.9. *Let R be a commutative ring with unity and $f=\prod_{i=1}^m(1+\xi_i t)$, $g=\prod_{j=1}^n(1+\eta_j t)$ polynomials with $\xi_i,\eta_j\in R$. Then we define*

$$f\otimes g:=\prod_{i=1}^m\prod_{j=1}^n(1+\xi_i\eta_j t).$$

The tensor product $f\otimes g$ can also be defined in the case where f and g do not factorize as in Definition 1.9, but we will not need this. In the setting of Definition 1.8, we have

$$\lambda_t(U\otimes V)\equiv\lambda_t(U)\otimes\lambda_t(V) \mod t^p \tag{1.2}$$

with $p := \mathrm{char}(K) > 0$ and U another KS-module (see [8, Theorem 1.16]; again, if the characteristic of K does not divide $|S|$, the above congruence becomes an equality). A third rule, which is elementary, is given by

$$\lambda_t(U \oplus V) = \lambda_t(U) \cdot \lambda_t(V) \quad \text{and} \quad \sigma_t(U \oplus V) = \sigma_t(U) \cdot \sigma_t(V). \tag{1.3}$$

We return to the situation where G is a group with normal Sylow p-subgroup of order p. Recall that we have made the assumption that the character $\alpha \in \mathrm{Irr}(H)$ given by conjugation has a square root $\beta \in \mathrm{Irr}(H)$ (see page 4), and that we have extended the representation ring R_{KG} by an element μ with $V_2 = \beta(\mu + \mu^{-1})$.

Theorem 1.10. *For $0 \leq n < p$ we have*

$$\lambda_t(V_{n+1}) = \prod_{j=0}^{n} (1 + \beta^n \mu^{n-2j} t).$$

Proof. Since V_1 is the trivial module, we have $\lambda_t(V_1) = V_1 + V_1 t = 1 + t$. For V_2 we have $\Lambda^1(V_2) = V_2 = \beta(\mu + \mu^{-1})$, and $\Lambda^2(V_2)$ is one-dimensional (since $\dim(V_2) = 2$). A $\tau \in H$ acts on $\Lambda^2(V_2)$ by multiplication with the determinant of the action on V_2. By Proposition 1.1, this determinant is $1 \cdot \alpha(\tau) = \alpha(\tau)$. Thus

$$\lambda_t(V_2) = 1 + \beta(\mu + \mu^{-1})t + \alpha t^2 = (1 + \beta\mu t)(1 + \beta\mu^{-1} t).$$

For $1 < n < p$, Lemma 1.2 yields $V_{n+1} \oplus V_\alpha \otimes V_{n-1} \cong V_2 \otimes V_n$. Using (1.2) and (1.3), we obtain

$$\lambda_t(V_{n+1}) \cdot (\lambda_t(V_\alpha) \otimes \lambda_t(V_{n-1})) \equiv \lambda_t(V_2) \otimes \lambda_t(V_n) \mod t^p$$

and so, by induction on n

$$\lambda_t(V_{n+1}) \cdot \left((1 + \alpha t) \otimes \prod_{j=0}^{n-2} (1 + \beta^{n-2} \mu^{n-2-2j} t) \right) \equiv \\ \big((1 + \beta\mu t)(1 + \beta\mu^{-1} t)\big) \otimes \prod_{j=0}^{n-1} (1 + \beta^{n-1} \mu^{n-1-2j} t) \mod t^p,$$

which, using Definition 1.9 twice, becomes

$$\lambda_t(V_{n+1}) \cdot \prod_{j=0}^{n-2} (1 + \beta^n \mu^{n-2-2j} t) \equiv \prod_{j=0}^{n-1} (1 + \beta^n \mu^{n-2j} t) \prod_{j=0}^{n-1} (1 + \beta^n \mu^{n-2-2j} t) \mod t^p.$$

Since the $(1 + \beta^n \mu^{n-2-2j} t)$ are invertible in $R_{KG}[\mu][[t]]$, we obtain

$$\lambda_t(V_{n+1}) \equiv \prod_{j=0}^{n} (1 + \beta^n \mu^{n-2j} t) \mod t^p.$$

Both sides of this congruence are polynomials of degree $n+1$, hence we have equality if $n < p-1$. It remains to compare the coefficients of t^p for $n = p-1$. G acts on $\Lambda^p(V_p)$ by the determinant, which is $\alpha^{p(p-1)/2}$ by Proposition 1.1. The coefficient of t^p on the right is $\beta^{(p-1)p}$, so we have equality. □

Corollary 1.11. *For $0 \leq n < p$ we have*

$$\sigma_t(V_{n+1}) \equiv \prod_{j=0}^{n} (1 - \beta^n \mu^{n-2j} t)^{-1} \mod t^p.$$

Proof. This follows from Theorem 1.10 and (1.1). $\square$

Corollary 1.11 only gives the symmetric powers of degrees less than p. This shortcoming was compensated for by Almkvist and Fossum [1] (see also [8, Section 2.3]) by proving a periodicity result for the symmetric powers. A similar approach can be taken in our case. In the symmetric algebra $K[V_n] = S^*(V_n) := \bigoplus_{r=0}^{\infty} S^r(V_n)$ we form the product $N := \prod_{\sigma \in P} \sigma(v_n)$, where $v_n = 1 + KP(\pi-1)^n \in V_n = KP/KP(\pi-1)^n$. Then $N \in K[V_n]^P$, and for $\tau \in H$ we have

$$\tau(N) = \prod_{\sigma \in P} \tau\sigma(v_n) = \prod_{\sigma \in P} \tau\sigma\tau^{-1}\tau(v_n) = N,$$

since $\tau(v_n) = v_n$. Thus $N \in K[V_n]^G$. Let $B_n \subset K[V_n]$ be the set of elements of $K[V_n]$ which, considered as polynomials in v_n, have degree less than p. B_n is a submodule of $K[V_n]$, and we have

$$K[V_n] = N \cdot K[V_n] \oplus B_n \tag{1.4}$$

(see [8, Lemma 2.9]; the proof uses division by N with remainder). Since N is an invariant, we have $K[V_n] \cong N \cdot K[V_n]$ as KG-modules. By Lemma 2.10 of [8], the homogeneous part $B_{n,r}$ of degree r is free as a KP-module for $r > p - n$, hence it is projective as a KG-module. Restricting Equation (1.4) to the part of degree $r + p$, we obtain:

Proposition 1.12. *For $1 \le n \le p$ and $r \ge 0$ we have*

$$S^{r+p}(V_n) \cong S^r(V_n) \oplus (\text{a projective } KG\text{-module}).$$

We can now apply the species coefficient-wise to $\sigma_t(V_{n+1})$.

Proposition 1.13. *Let $\phi_{\tau,\gamma}$ be a species of G over K as given in Theorem 1.6, and let $0 \le n < p$. Then we have*

(a) If $\phi_{\tau,\gamma}$ is p-regular, i.e., $\gamma = \beta(\tau)^{\pm 1}$, then

$$\phi_{\tau,\gamma}\left(\sigma_t(V_{n+1})\right) = \prod_{j=0}^{n} (1 - \alpha(\tau)^j t)^{-1}.$$

(b) If $\phi_{\tau,\gamma}$ is p-singular, i.e., γ is a $(2p)$-th root of unity, $\gamma \ne \pm 1$, then

$$\phi_{\tau,\gamma}\left(\sigma_t(V_{n+1})\right) = \frac{1 - (\beta(\tau)^n \gamma^{-n} t)^p}{1 - t^p} \prod_{j=0}^{n} (1 - \beta(\tau)^n \gamma^{n-2j} t)^{-1}.$$

Notice that for $\gamma = \beta(\tau)$ these formulas coincide.

Proof. If $\phi_{\tau,\gamma}$ is p-regular, then by Remark 1.7 $\phi_{\tau,\gamma}$ is the evaluation of the Brauer character at τ. By Proposition 1.1, the eigenvalues of the action of τ in V_{n+1} lift to $1, \alpha(\tau), \ldots, \alpha(\tau)^n$. The formula given in (a) follows easily (see Proposition 2.1(b)).

The proof of (b) is as the proof of Proposition 3.1 in [8]. The main observation is that $\phi_{\tau,\gamma}$ vanishes on a projective module $V_{p,\chi}$ by Lemma 1.3. Thus by Proposition 1.12, $\phi_{\tau,\gamma}\left(S^r(V_{n+1})\right)$ is periodic in r with period p. For more details we refer to [8]. $\square$

We finish this section by determining the application of the species to $\sigma_t(V_{n+1,\chi})$.

Theorem 1.14. *For $\chi \in \mathrm{Irr}(H)$ choose a matrix representation $M_\chi\colon H \to \mathrm{GL}_{\chi(1)}(\mathbb{C})$ whose character is χ. For $0 \le n < p$, $\tau \in H$ and $\gamma \in M_{2p} \cup \{\beta(\tau)\}$, set*

$$M_{n+1,\chi}(\gamma,\tau) := \beta(\tau)^n \cdot \begin{pmatrix} \gamma^n & & & & \\ & \gamma^{n-2} & & & \\ & & \ddots & & \\ & & & \gamma^{-(n-2)} & \\ & & & & \gamma^{-n} \end{pmatrix} \otimes M_\chi(\tau) \in \mathrm{GL}_{(n+1)\chi(1)}(\mathbb{C}).$$

Then we have

$$\phi_{\tau,\gamma}\left(\sigma_t(V_{n+1,\chi})\right) = \frac{\det\left(1 - (\beta(\tau)^n \gamma^{-n} t \cdot M_\chi(\tau))^p\right)}{\det\left(1 - (t \cdot M_\chi(\tau))^p\right)} \det\left(1 - t \cdot M_{n+1,\chi}(\gamma,\tau)\right)^{-1}.$$

Here $\phi_{\tau,\gamma}$ is the species introduced in Theorem 1.6.

Proof. We will prove the theorem without assuming that χ is an irreducible character. Let $G_0 := \langle \tau, P\rangle$ be the subgroup generated by τ and P. Then for $r \ge 0$ we have a commutative diagram

$$\begin{array}{ccccc} R_{KG} & \xrightarrow{S^r} & R_{KG} & \xrightarrow{\phi_{\tau,\gamma}} & \mathbb{C} \\ \big\downarrow \mathrm{res}_{G,G_0} & & \big\downarrow \mathrm{res}_{G,G_0} & & \big\downarrow \mathrm{id} \\ R_{KG_0} & \xrightarrow{S^r} & R_{KG_0} & \xrightarrow{\phi_{\tau,\gamma}} & \mathbb{C}, \end{array}$$

where S^r stands for taking the r-th symmetric power. Thus we may replace G by G_0, i.e., we assume that H is cyclic. Observe that both sides of the claimed equation, considered as functions $f(\chi)$ of χ, satisfy the rule $f(\chi_1 + \chi_2) = f(\chi_1) \cdot f(\chi_2)$. Thus we may assume that χ is irreducible and hence one-dimensional. But then it is elementary to see that

$$\sigma_t(V_{n+1,\chi}) = \sigma_{\chi \cdot t}(V_{n+1}).$$

By Proposition 1.13, the right hand side of the claimed equality is $\phi_{\tau,\gamma}(\sigma_{\chi\cdot t}(V_{n+1}))$. This completes the proof. ◻

1.3 Averaging operators and Hilbert series

In this section we assume the situation introduced at the beginning of Section 1. We also assume that the character α given by the conjugation action of H on P has a square root $\beta \in \mathrm{Irr}(H)$ (see page 4). In the previous section we have derived formulas which yield the application of the species on the sigma-series $\sigma_t(V_{n+1,\chi})$. We want to find expressions giving the Hilbert series of the invariant ring and generating functions encoding the multiplicities of indecomposable summands in the symmetric powers of a KG-module. For this purpose we use weighted averages over the species. Since there are as many species as there are indecomposable KG-modules (see Remark 1.5) and since they are linearly independent, one can get every $\mathbb{Z}$-linear map $R_{KG} \to \mathbb{C}$ as a unique $\mathbb{C}$-linear combination of the species. The map $V \mapsto \dim(V^G)$ is such a map. The only question is how one can find a general expression giving the coefficients of this (and other) linear maps. We remind the reader that $M_{2p} \subset \mathbb{C}$ is the set of $(2p)$-th roots of unity, and the p-singular species are given as $\phi_{\tau,\gamma}$ with $\tau \in H$ and $\gamma \in M_{2p} \setminus \{\pm 1\}$ by Theorem 1.6. Moreover, for each $\rho \in G_{p'}$ (the set of elements of order coprime to p) we have a p-regular species ch_ρ, which assigns to a KG-module V the Brauer character of V evaluated at ρ. The p-regular species can also be described as $\phi_{\tau,\beta(\tau)^{\pm 1}}$ where $\tau \in H$ is conjugate to ρ (see Remark 1.7). The indecomposables are given as $V_{n,\chi}$ with $1 \le n \le p$ and $\chi \in \mathrm{Irr}(H)$ by Proposition 1.1.

Proposition 1.15. *For $\chi \in \operatorname{Irr}(H)$, define operators $R_{KG} \to \mathbb{C}$ as follows:*

$$T_{n,\chi} := \frac{1}{2|G|} \sum_{\tau \in H} \sum_{\substack{\gamma \in M_{2p},\\ \gamma \neq \pm 1}} \chi(\tau^{-1})\beta(\tau^{-1})^{n-1}\gamma^{-n}(\gamma - \gamma^{-1}) \cdot \phi_{\tau,\gamma} \quad (1 \leq n < p), \quad \textit{and}$$

$$T_{p,\chi} := \frac{1}{|G|}\left(\sum_{\rho \in G_{p'}} \chi(\rho^{-1}) \cdot \operatorname{ch}_\rho - \frac{1}{4} \sum_{\tau \in H} \sum_{\substack{\gamma \in M_{2p},\\ \gamma \neq \pm 1}} \frac{\chi(\tau^{-1})(\gamma - \gamma^{-1})^2 (\beta(\tau)\gamma)^p}{(1 - \beta(\tau)\gamma)(1 - \beta(\tau)\gamma^{-1})} \cdot \phi_{\tau,\gamma} \right).$$

Furthermore, define

$$T := \frac{1}{|G|}\left(\sum_{\rho \in G_{p'}} \operatorname{ch}_\rho - \frac{1}{4} \sum_{\tau \in H} \sum_{\substack{\gamma \in M_{2p},\\ \gamma \neq \pm 1}} \frac{(\gamma - \gamma^{-1})^2}{(1 - \beta(\tau)\gamma)(1 - \beta(\tau)\gamma^{-1})} \cdot \phi_{\tau,\gamma} \right).$$

Then for $1 \leq k \leq p$ and $\psi \in \operatorname{Irr}(H)$ we have

$$T_{n,\chi}(V_{k,\psi}) = \delta_{n,k}\delta_{\chi,\psi}. \tag{1.5}$$

Moreover, for any KG-modules V we have

$$T(V) = \dim(V^G). \tag{1.6}$$

Proof. It is straightforward to verify Equation (1.5) for $n < p$ by using Lemma 1.3 and character theory (see Proposition 3.2 in [8]).

To prove Equation (1.5) for $n = p$, we claim that

$$T_{p,\chi} = U_\chi - \sum_{n=1}^{p-1} \sum_{\psi \in \operatorname{Irr}(H)} U_\chi(V_{n,\psi}) T_{n,\psi}, \tag{1.7}$$

where

$$U_\chi := \frac{1}{|G|}\left(\sum_{\rho \in G_{p'}} \chi(\rho^{-1}) \cdot \operatorname{ch}_\rho \right).$$

We first check that this would lead to Equation (1.5) for $n = p$. By orthogonality relations (see Feit [7, Chapter IV, Lemma 3.3]) we have $U_\chi(V_{p,\psi}) = \delta_{\chi,\psi}$. Since $\phi_{\tau,\gamma}(V_{p,\psi}) = 0$ for $\gamma \in M_{2p} \setminus \{\pm 1\}$, Equation (1.7) leads to

$$T_{p,\chi}(V_{p,\psi}) = U_\chi(V_{p,\psi}) = \delta_{\chi,\psi}.$$

On the other hand, for $k < p$, (1.7) and Equation (1.5) for $n < p$ yield

$$T_{p,\chi}(V_{k,\psi}) = U_\chi(V_{k,\psi}) - U_\chi(V_{k,\psi}) = 0.$$

Thus we are done if we can prove (1.7). In view of the definition of $T_{p,\chi}$ we have to show that

$$\sum_{n=1}^{p-1} \sum_{\psi \in \operatorname{Irr}(H)} U_\chi(V_{n,\psi}) T_{n,\psi} = \frac{1}{4|G|} \sum_{\tau \in H} \sum_{\substack{\gamma \in M_{2p},\\ \gamma \neq \pm 1}} \frac{\chi(\tau^{-1})(\gamma - \gamma^{-1})^2 (\beta(\tau)\gamma)^p}{(1 - \beta(\tau)\gamma)(1 - \beta(\tau)\gamma^{-1})} \cdot \phi_{\tau,\gamma}. \tag{1.8}$$

An elementary calculation shows that every $\rho \in G_{p'}$ is conjugate to a $\tau \in H$. Moreover, if $\tau \in H$ lies in the centralizer $\mathcal{C}_H(P)$, then the number of G-conjugates of τ is equal to the number $|\tau^H|$ of H-conjugates of τ. On the other hand, if $\tau \in H$ but $\tau \notin \mathcal{C}_H(P)$, then $|\tau^G| = p \cdot |\tau^H|$. Thus the summation over $\rho \in G_{p'}$ in the definition of U_χ is essentially a summation over $\tau \in H$. For $\tau \in H$ we have

$$\sum_{n=1}^{p-1} \sum_{\psi \in \mathrm{Irr}(H)} \mathrm{ch}_\tau(V_{n,\psi}) T_{n,\psi} = \frac{1}{2|G|} \sum_{n=1}^{p-1} \sum_{\psi \in \mathrm{Irr}(H)} \sum_{\sigma \in H} \sum_{\gamma \in M_{2p}} \mathrm{ch}_\tau(V_{n,\psi}) \psi(\sigma^{-1}) \beta(\sigma^{-1})^{n-1} \gamma^{-n} (\gamma - \gamma^{-1}) \cdot \phi_{\sigma,\gamma}. \tag{1.9}$$

Since $\mathrm{ch}_\tau(V_{n,\psi}) = \psi(\tau)\,\mathrm{ch}_\tau(V_n)$, we obtain

$$\sum_{n=1}^{p-1} \sum_{\psi \in \mathrm{Irr}(H)} \mathrm{ch}_\tau(V_{n,\psi}) \psi(\sigma^{-1}) \beta(\sigma^{-1})^{n-1} \gamma^{-n} = \left(\sum_{\psi \in \mathrm{Irr}(H)} \psi(\tau)\psi(\sigma^{-1}) \right) \left(\sum_{n=1}^{p-1} \mathrm{ch}_\tau(V_n) \beta(\sigma^{-1})^{n-1} \gamma^{-n} \right). \tag{1.10}$$

Using character relations, we see that the first sum in the above product evaluates to $|\mathcal{C}_H(\tau)|$ if σ is conjugate to τ in H, and 0 otherwise. In the case $\sigma \sim_H \tau$, the second sum is essentially a geometric sum, which evaluates to

$$\frac{\gamma^p}{(1-\beta(\tau)\gamma)(1-\beta(\tau)\gamma^{-1})} \cdot \begin{cases} (\beta(\tau)^p \gamma - \beta(\tau)\gamma^p) & \text{if } \tau \in H \setminus \mathcal{C}_H(P), \\ (1 - p + p\beta(\tau)\gamma - \beta(\tau)\gamma^p) & \text{if } \tau \in \mathcal{C}_H(P),\ \gamma \notin \{\pm 1\}. \end{cases}$$

(Observe that $\gamma^p = \pm 1$ and $\beta(\tau)^p = \pm\beta(\tau)$.) Substituting this into (1.10) and (1.9), and adding the same expression with γ substituted by γ^{-1} (which does not change the species), we obtain

$$\sum_{n=1}^{p-1} \sum_{\psi \in \mathrm{Irr}(H)} \mathrm{ch}_\tau(V_{n,\psi}) T_{n,\psi} = \frac{|H|}{4|G|} \sum_{\substack{\gamma \in M_{2p}, \\ \gamma \neq \pm 1}} \frac{\gamma^p (\gamma - \gamma^{-1})^2 \beta(\tau)^p}{(1-\beta(\tau)\gamma)(1-\beta(\tau)\gamma^{-1})} \cdot \left\{ \begin{array}{ll} 1, & \tau \notin \mathcal{C}_H(P) \\ p, & \tau \in \mathcal{C}_H(P) \end{array} \right\} \cdot \phi_{\tau,\gamma}.$$

We can now evaluate $\sum_{n=1}^{p-1} \sum_{\psi \in \mathrm{Irr}(H)} U_\chi(V_{n,\psi}) T_{n,\psi}$ and obtain (1.8). This completes the proof of Equation (1.5).

To prove Equation (1.6), observe that by Proposition 1.1 we have $\dim(V_{n,\chi}^G) = \delta_{\chi,\beta^{-2(n-1)}}$. Therefore

$$T := \sum_{n=1}^{p} T_{n,\beta^{-2(n-1)}} \tag{1.11}$$

satisfies (1.6). We have

$$\sum_{n=1}^{p-1} T_{n,\beta^{-2(n-1)}} = \frac{1}{2|G|} \sum_{\tau \in H} \sum_{\gamma \in M_{2p}} \sum_{n=1}^{p-1} \beta(\tau)^{n-1} \gamma^{-n} (\gamma - \gamma^{-1}) \cdot \phi_{\tau,\gamma}$$

Evaluating the geometric sum and adding the same expression with γ substituted by γ^{-1} yields

$$\sum_{n=1}^{p-1} T_{n,\beta^{-2(n-1)}} = \frac{1}{4|G|} \sum_{\tau \in H} \sum_{\substack{\gamma \in M_{2p}, \\ \gamma \neq \pm 1}} \frac{(\gamma - \gamma^{-1})^2 (\beta(\tau)^p \gamma^p - 1)}{(1-\beta(\tau)\gamma)(1-\beta(\tau)\gamma^{-1})} \phi_{\tau,\gamma}. \tag{1.12}$$

Substituting this into (1.11) yields Equation (1.6). $\square$

Definition 1.16. *For a finite-dimensional KG-module V and an indecomposable KG-module U we write $m_U(V)$ for the multiplicity of U in a decomposition of V as a direct sum of indecomposables. Then we define*

$$H_U(K[V],t) := \sum_{r=0}^{\infty} m_U(S^r(V)) \cdot t^r \in \mathbb{C}[[t]],$$

and

$$H(K[V]^G,t) := \sum_{r=0}^{\infty} \dim(S^r(V)^G) \cdot t^r \in \mathbb{C}[[t]].$$

$H(K[V]^G,t)$ is the Hilbert series of the invariant ring. We call $H_U(K[V],t)$ a decomposition series of $K[V]$.

We can now put everything together and give formulas for $H_W(K[V],t)$ and $H(K[V]^G,t)$.

Theorem 1.17. *With the hypotheses and notation of Proposition 1.15, let V be a finite-dimensional KG-module. Suppose that V has the decomposition*

$$V = V_{n_1,\chi_1} \oplus \cdots \oplus V_{n_k,\chi_k}$$

into indecomposable KG-modules (see Proposition 1.1). Then for a species $\phi_{\tau,\gamma}$ with $\tau \in H$ and $\gamma = \beta(\tau)$ or $\gamma \in M_{2p} \setminus \{\pm 1\}$ a $(2p)$-th root of unity, we have

$$\phi_{\tau,\gamma}\left(\sigma_t(V)\right) = \phi_{\tau,\gamma}\left(\sigma_t(V_{n_1,\chi_1})\right) \cdots \phi_{\tau,\gamma}\left(\sigma_t(V_{n_k,\chi_k})\right),$$

where the $\phi_{\tau,\gamma}\left(\sigma_t(V_{n_i,\chi_i})\right)$ are given in Theorem 1.14. Moreover, for $\rho \in G_{p'}$ with $\rho \sim_G \tau \in H$ we have $\mathrm{ch}_\rho = \phi_{\tau,\beta(\tau)}$. For $\chi \in \mathrm{Irr}(H)$, the following formulas hold:

$$H_{V_{n,\chi}}(K[V],t) = \frac{1}{2|G|} \sum_{\tau \in H} \sum_{\gamma \in M_{2p}} \chi(\tau^{-1})\beta(\tau^{-1})^{n-1}\gamma^{-n}(\gamma - \gamma^{-1}) \cdot \phi_{\tau,\gamma}\left(\sigma_t(V)\right) \quad (1 \le n < p),$$

$$H_{V_{p,\chi}}(K[V],t) := \frac{1}{|G|}\left(\sum_{\rho \in G_{p'}} \chi(\rho^{-1}) \cdot \mathrm{ch}_\rho\left(\sigma_t(V)\right) - \frac{1}{4}\sum_{\tau \in H} \sum_{\substack{\gamma \in M_{2p},\\ \gamma \ne \pm 1}} \frac{\chi(\tau^{-1})(\gamma - \gamma^{-1})^2(\beta(\tau)\gamma)^p}{(1-\beta(\tau)\gamma)(1-\beta(\tau)\gamma^{-1})} \cdot \phi_{\tau,\gamma}\left(\sigma_t(V)\right)\right).$$

Moreover,

$$H(K[V]^G,t) = \frac{1}{|G|}\left(\sum_{\rho \in G_{p'}} \mathrm{ch}_\rho\left(\sigma_t(V)\right) - \frac{1}{4}\sum_{\tau \in H} \sum_{\substack{\gamma \in M_{2p},\\ \gamma \ne \pm 1}} \frac{(\gamma - \gamma^{-1})^2}{(1-\beta(\tau)\gamma)(1-\beta(\tau)\gamma^{-1})} \cdot \phi_{\tau,\gamma}\left(\sigma_t(V)\right)\right).$$

2 Groups with a Sylow p-subgroup of order p

In this section we suppose that G is a finite group with a Sylow p-subgroup P of order p, but we do not assume that P is normal in G. Moreover, K is a splitting field of G with $\mathrm{char}(K) = p$. Let $N := \mathcal{N}_G(P)$ be the normalizer of P in G, and let H be a complement of P in N, so $N \cong P \rtimes H$. As on page 4, if necessary we choose an extension $\widetilde{N} \to N$ whose kernel is of order at most 2, such that the Brauer character α given by the conjugation action of H on P has a square root β. We write $\widetilde{H}$ for the preimage of H in $\widetilde{N}$. Then $\widetilde{N} = P \rtimes \widetilde{H}$ and $\beta \in \mathrm{Irr}(\widetilde{H})$.

Since P is a trivial intersection subgroup, Theorem 1 on page 71 of Alperin [2] gives a bijective correspondence between the non-projective indecomposable KG-modules and the non-projective indecomposable KN-modules. This correspondence is a special case of the Green correspondence, and it is given as follows. To a non-projective indecomposable KG-module V we associate the non-projective indecomposable KN-module U with

$$V_N \cong U \oplus (\text{projective}),$$

where V_N is the restriction. On the other hand, to a non-projective indecomposable KN-module U we associate the non-projective indecomposable KG-module V with

$$U^{\uparrow G} \cong V \oplus (\text{projective}),$$

where $U^{\uparrow G}$ is the induced module. It is shown in [2, p. 71] that V_N and $U^{\uparrow G}$ have decompositions of the above forms.

2.1 Species and decomposition series

The homomorphisms $\widetilde{N} \to N$ and $N \hookrightarrow G$ induce ring-homomorphisms $R_{KN} \to R_{K\widetilde{N}}$ and $R_{KG} \to R_{KN}$. For a p-singular species $\phi_{\tau,\gamma}$ of $\widetilde{N}$ with $\tau \in \widetilde{H}$, $\gamma \in M_{2p} \setminus \{\pm 1\}$ as given in Theorem 1.6, the composition

$$R_{KG} \to R_{KN} \to R_{K\widetilde{N}} \xrightarrow{\phi_{\tau,\gamma}} \mathbb{C} \tag{2.1}$$

gives a species of G. From Lemma 1.3 and the definition of $\phi_{\tau,\gamma}$ it is clear that this composition remains the same if one changes τ by the non-trivial element of $\ker(\widetilde{H} \to H)$ and substitutes γ by $-\gamma$. Therefore the number of species obtained in this way coincides with the number of non-projective indecomposable KG-modules. Let us denote the species given by (2.1) by $\Phi_{\tau,\gamma}$.

In addition, we have the p-regular species ch_ρ given by evaluating the Brauer character of a KG-module at a p-regular element $\rho \in G_{p'}$. The ch_ρ give as many p-regular species as there are projective indecomposables. Since all $\Phi_{\tau,\gamma}$ vanish on the projective modules and by orthogonality relations none of the p-regular species do, we conclude by counting and Remark 1.5 that we have described all species of G over K. We say that the $\Phi_{\tau,\gamma}$ are p-singular species. We can give the evaluation of the species on the sigma-series (see Definition 1.8) of a KG-module.

Proposition 2.1. *Let V be a finite-dimensional KG-module.*

(a) Suppose that the restriction V_N to N has the decomposition

$$V_N \cong V_{n_1,\chi_1} \oplus \cdots \oplus V_{n_k,\chi_k}$$

into indecomposables (see Proposition 1.1). Then for $\tau \in \widetilde{H}$ and $\gamma \in M_{2p} \setminus \{\pm 1\}$ we have

$$\Phi_{\tau,\gamma}\left(\sigma_t(V)\right) = \phi_{\tau,\gamma}\left(\sigma_t(V_{n_1,\chi_1})\right) \cdots \phi_{\tau,\gamma}\left(\sigma_t(V_{n_k,\chi_k})\right),$$

where $\phi_{\tau,\gamma}\left(\sigma_t(V_{n_i,\chi_i})\right)$ are given by Theorem 1.14.

(b) Let $\rho \in G_{p'}$ be a p-regular element, and let $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ be liftings of the eigenvalues of ρ acting on V (with respect to the p-modular system (K_0, R, K)). Then

$$\mathrm{ch}_\rho\left(\sigma_t(V)\right) = \frac{1}{(1-\lambda_1 t)\cdots(1-\lambda_n t)}.$$

Proof. Since $S^r(V)_N \cong S^r(V_N)$, we have $\Phi_{\tau,\gamma}\left(\sigma_t(V)\right) = \phi_{\tau,\gamma}\left(\sigma_t(V_N)\right)$, so part (a) follows from (1.3). For the proof of (b) we may assume that $G = \langle \rho \rangle$ and V is indecomposable (see the proof of Theorem 1.14). But then $\dim(V) = 1$ and (b) is clear. $\square$

We can also give averaging operators as in Proposition 1.15, but in a less explicit form. For a non-projective indecomposable KN-module U, denote its Green-correspondent by $\Theta(U)$, so $U^{\uparrow G} \cong \Theta(U) \oplus (\text{projective})$. Thus the non-projective indecomposable KG-modules are precisely the $\Theta(V_{n,\chi})$ with $1 \leq n < p$ and $\chi \in \mathrm{Irr}(H)$. On the other hand, the projective indecomposable KG-modules are determined by their socles, which are simple (see Alperin [2, p. 41]). We denote the projective indecomposable KG-module whose socle has the irreducible Brauer character φ by V_φ. In Propositions 2.2 and 2.3 we extend Proposition 1.15 to G.

Proposition 2.2. *For $\chi \in \mathrm{Irr}(H)$ and $1 \leq n < p$, define the operator*

$$T_{n,\chi} := \frac{1}{2|\widetilde{N}|} \sum_{\tau \in \widetilde{H}} \sum_{\gamma \in M_{2p}} \chi(\tau^{-1}) \beta(\tau^{-1})^{n-1} \gamma^{-n} (\gamma - \gamma^{-1}) \cdot \Phi_{\tau,\gamma}$$

on R_{KG}. For an irreducible Brauer character $\varphi \in \mathrm{IBr}_K(G)$, define

$$U_\varphi := \frac{1}{|G|} \sum_{\rho \in G_{p'}} \varphi(\rho^{-1}) \operatorname{ch}_\rho.$$

Furthermore, define

$$T_\varphi := U_\varphi - \sum_{n=1}^{p-1} \sum_{\chi \in \mathrm{Irr}(H)} U_\varphi(\Theta(V_{n,\chi})) \cdot T_{n,\chi}.$$

Then we have

$$\begin{aligned} T_{n,\chi}(\Theta(V_{k,\eta})) &= \delta_{n,k} \cdot \delta_{\chi,\eta} && \text{for} \quad 1 \leq k < p,\ \eta \in \mathrm{Irr}(H), \\ T_{n,\chi}(V_\varphi) &= 0 && \text{for} \quad \varphi \in \mathrm{IBr}_K(G), \\ T_\varphi(\Theta(V_{n,\chi})) &= 0 && \text{for} \quad 1 \leq n < p,\ \chi \in \mathrm{Irr}(H), \\ T_\varphi(V_\psi) &= \delta_{\varphi,\psi} && \text{for} \quad \psi \in \mathrm{IBr}_K(G). \end{aligned}$$

Remark. The formula for T_φ cannot be made completely explicit since more detailed knowledge of the modules $\Theta(V_{n,\chi})$ is necessary for the evaluation of $U_\varphi(\Theta(V_{n,\chi}))$. More precisely, we have $\Theta(V_{n,\chi}) = V_{n,\chi}^{\uparrow G} - Q$ with Q a projective KG-module. By Frobenius reciprocity (see Curtis and Reiner [5, Theorem 10.9]) we have

$$U_\varphi(V_{n,\chi}^{\uparrow G}) = \frac{1}{|N|} \sum_{\rho \in N_{p'}} \varphi(\rho^{-1}) \operatorname{ch}_\rho(V_{n,\chi}),$$

where $\operatorname{ch}_\rho(V_{n,\chi})$ is explicitly known. From this we have to subtract $U_\varphi(Q)$, which by orthogonality relations is the multiplicity of V_φ in Q. Thus the decomposition of Q has to be known in order to get an explicit formula for T_φ. $\triangleleft$

Proof of Proposition 2.2. If V is projective, then V_N is also projective, so $\Phi_{\tau,\gamma}(V) = \phi_{\tau,\gamma}(V_N) = 0$. This yields the second formula. It also follows that

$$T_{n,\chi}\left(\Theta(V_{k,\eta})\right) = T_{n,\chi}\left(V_{k,\eta}^{\uparrow G}\right) = T_{n,\chi}\left((V_{k,\eta}^{\uparrow G})_N\right),$$

where the last $T_{n,\chi}$ denotes the operator defined in Proposition 1.15. By Green-correspondence $(V_{k,\eta}^{\uparrow G})_N \cong V_{k,\eta} \oplus (\text{projective})$, hence

$$T_{n,\chi}\left((\Theta(V_{k,\eta})\right) = T_{n,\chi}(V_{k,\eta}) = \delta_{n,k} \cdot \delta_{\chi,\eta}$$

by Proposition 1.15. This yields the first formula.

For $\psi \in \mathrm{IBr}_K(G)$ we have $U_\varphi(V_\psi) = \delta_{\varphi,\psi}$ by orthogonality relations. Since $T_{n,\chi}(V_\psi) = 0$, this implies the fourth formula. Finally,

$$T_\varphi(\Theta(V_{n,\chi})) = U_\varphi(\Theta(V_{n,\chi})) - U_\varphi(\Theta(V_{n,\chi})) = 0.$$

$\square$

By putting Propositions 2.1 and 2.2 together, we obtain formulas for the decomposition series $H_U(K[V], t)$.

2.2 The Hilbert series

We have an easier game if we want to compute the Hilbert series of the invariant ring instead of the decomposition series. Indeed, by Frobenius reciprocity (see Benson [3, Proposition 3.3.1]), we have

$$\dim(U^{\uparrow G})^G = \dim(U^N)$$

for a KN-module U. In particular,

$$\dim(V_{n,\chi}^{\uparrow G})^G = \delta_{\chi,\alpha^{-(n-1)}} \tag{2.2}$$

(see before Equation (1.11)). This means that we do not need any information on how $V_{n,\chi}^{\uparrow G}$ decomposes into $\Theta(V_{n,\chi})$ and a projective module. Moreover, for the projective indecomposable KG-modules V_φ we have

$$\dim(V_\varphi^G) = \delta_{1,\varphi},$$

since V_φ has a simple socle with Brauer character φ.

Proposition 2.3. *Define the operator*

$$T := \frac{1}{|G|} \sum_{\rho \in G_{p'}} \mathrm{ch}_\rho - \frac{1}{4p|\widetilde{H}|} \sum_{\tau \in \widetilde{H}} \sum_{\substack{\gamma \in M_{2p}, \\ \gamma \neq \pm 1}} \frac{(\gamma - \gamma^{-1})^2}{(1 - \beta(\tau)\gamma)(1 - \beta(\tau)\gamma^{-1})} \cdot \Phi_{\tau,\gamma}.$$

Here $G_{p'}$ is the set of p-regular elements in G, $M_{2p} \subset \mathbb{C}$ is the set of $(2p)$-th roots of unity, and the species ch_ρ and $\Phi_{\tau,\gamma}$ were introduced at the beginning of Section 2.1. Then for a KG-module V we have

$$T(V) = \dim(V^G).$$

Proof. Set

$$U := \frac{1}{|G|} \sum_{\rho \in G_{p'}} \mathrm{ch}_\rho\,.$$

Then by orthogonality relations we have $U(V_\varphi) = \delta_{1,\varphi}$, and so if V is projective then $U(V) = \dim(V^G)$. Since the $\Phi_{\tau,\gamma}$ vanish on projective modules, it follows that

$$T(V_\varphi) = \delta_{1,\varphi} = \dim(V_\varphi^G).$$

Let $T_{n,\chi}$ be the operators from Proposition 2.2 ($1 \leq n < p$, $\chi \in \mathrm{Irr}(H)$), and set

$$T_0 := \sum_{n=1}^{p-1} T_{n,\alpha^{-(n-1)}}.$$

Then for a non-projective indecomposable KG-module $\Theta(V_{n,\chi})$ we have by Proposition 2.2 and (2.2)

$$T_0\left(\Theta(V_{n,\chi})\right) = \delta_{\chi,\alpha^{-(n-1)}} = \dim(V_{n,\chi}^{\uparrow G})^G. \tag{2.3}$$

We claim that

$$T = U + T_0 - \sum_{n=1}^{p-1} \sum_{\chi \in \mathrm{Irr}(H)} U(V_{n,\chi}^{\uparrow G}) \cdot T_{n,\chi}. \tag{2.4}$$

This would imply the theorem, since (2.3), (2.4), and Proposition 2.2 yield

$$\begin{aligned} T\left(\Theta(V_{n,\chi})\right) &= U\left(\Theta(V_{n,\chi})\right) + \dim(V_{n,\chi}^{\uparrow G})^G - U\left(V_{n,\chi}^{\uparrow G}\right) = \dim(V_{n,\chi}^{\uparrow G})^G - U\left(V_{n,\chi}^{\uparrow G} - \Theta(V_{n,\chi})\right) \\ &= \dim(V_{n,\chi}^{\uparrow G})^G - \dim\left(V_{n,\chi}^{\uparrow G} - \Theta(V_{n,\chi})\right)^G = \dim\left(\Theta(V_{n,\chi})^G\right), \end{aligned}$$

since $V_{n,\chi}^{\uparrow G} - \Theta(V_{n,\chi})$ is projective. For the proof of (2.4), observe that by Frobenius reciprocity (see Curtis and Reiner [5, Theorem 10.9]) we have

$$U(V_{n,\chi}^{\uparrow G}) = \frac{1}{|N|} \sum_{\rho \in N_{p'}} \operatorname{ch}_\rho(V_{n,\chi}).$$

Thus

$$\sum_{n=1}^{p-1} \sum_{\chi \in \operatorname{Irr}(H)} U(V_{n,\chi}^{\uparrow G}) \cdot T_{n,\chi} = \\ \frac{1}{2|N||\widetilde{N}|} \sum_{n=1}^{p-1} \sum_{\chi \in \operatorname{Irr}(H)} \sum_{\rho \in N_{p'}} \sum_{\tau \in \widetilde{H}} \sum_{\gamma \in M_{2p}} \operatorname{ch}_\rho(V_{n,\chi}) \chi(\tau^{-1}) \beta(\tau^{-1})^{n-1} \gamma^{-n} (\gamma - \gamma^{-1}) \cdot \Phi_{\tau,\gamma}. \tag{2.5}$$

Let $\pi\colon \widetilde{H} \to H$ be the restriction of $\widetilde{N} \to N$ to $\widetilde{H}$. As in the proof of Proposition 1.15 we see that the sum

$$\sum_{n=1}^{p-1} \sum_{\chi \in \operatorname{Irr}(H)} \operatorname{ch}_\rho(V_{n,\chi}) \chi(\tau^{-1}) \beta(\tau^{-1})^{n-1} \gamma^{-n} (\gamma - \gamma^{-1}) \cdot \Phi_{\tau,\gamma}$$

is only non-zero if $\pi(\tau)$ and ρP are conjugate in H, and in this case evaluates to

$$\frac{|\mathcal{C}_H(\pi(\tau))| \cdot (\beta(\tau)\gamma)^p (\gamma - \gamma^{-1})^2}{2(1 - \beta(\tau)\gamma)(1 - \beta(\tau)\gamma^{-1})} \cdot \begin{Bmatrix} 1 & \text{if } \tau \in \widetilde{H} \setminus \mathcal{C}_{\widetilde{H}}(P), \\ p & \text{if } \tau \in \mathcal{C}_{\widetilde{H}}(P),\ \gamma \notin \{\pm 1\} \end{Bmatrix} \cdot \Phi_{\tau,\gamma}.$$

It is easy to see that for $\tau \in \widetilde{H}$, the number of $\rho \in N_{p'}$ with $\rho P \sim_H \pi(\tau)$ is $p \cdot [H : \mathcal{C}_H(\pi(\tau))]$ if $\tau \notin \mathcal{C}_{\widetilde{H}}(P)$, and $[H : \mathcal{C}_H(\pi(\tau))]$ otherwise. Thus we obtain

$$\sum_{n=1}^{p-1} \sum_{\chi \in \operatorname{Irr}(H)} U(V_{n,\chi}^{\uparrow G}) \cdot T_{n,\chi} = \frac{1}{4|\widetilde{N}|} \sum_{\tau \in \widetilde{H}} \sum_{\substack{\gamma \in M_{2p}, \\ \gamma \neq \pm 1}} \frac{(\gamma - \gamma^{-1})^2 (\beta(\tau)\gamma)^p}{(1 - \beta(\tau)\gamma)(1 - \beta(\tau)\gamma^{-1})} \cdot \Phi_{\tau,\gamma}. \tag{2.6}$$

For T_0, we obtain as in Equation (1.12)

$$T_0 = \frac{1}{4|\widetilde{N}|} \sum_{\tau \in \widetilde{H}} \sum_{\substack{\gamma \in M_{2p}, \\ \gamma \neq \pm 1}} \frac{(\gamma - \gamma^{-1})^2 (\beta(\tau)^p \gamma^p - 1)}{(1 - \beta(\tau)\gamma)(1 - \beta(\tau)\gamma^{-1})} \Phi_{\tau,\gamma}.$$

Substituting this and Equation (2.6) into (2.4) yields the proposition. □

Putting Propositions 2.1 and 2.3 together, we have proved:

Theorem 2.4. *Let G be finite group and K a field of characteristic $p > 0$ such that $|G|$ is divisible by p but not by p^2. Then*

$$H(K[V]^G, t) = T\left(\sigma_t(V)\right),$$

where T is the operator defined in Proposition 2.3. The $\operatorname{ch}_\rho(\sigma_t(V))$ and $\Phi_{\tau,\gamma}(\sigma_t(V))$, which are needed for the evaluation of $T\left(\sigma_t(V)\right)$, are given in Proposition 2.1.

Define the degree of a rational function in t as the difference between the degree of the numerator and the degree of the denominator as polynomials in t. If S is a finite group and K a field of characteristic p not dividing $|G|$, then it is easily seen from Molien's theorem that for an n-dimensional KS-module V we have

$$\deg\left(H(K[V]^S,t)\right) \leq -n.$$

It has been conjectured that this upper bound still holds if we drop the assumption $p \nmid |G|$ (see Kemper [9, Conjecture 22]). By inspection of the expressions involved in Theorem 2.4 we obtain the following result.

Corollary 2.5. *Let G be a finite group and K a field of characteristic p such that $p^2 \nmid |G|$. Then for a finite-dimensional KG-module V we have*

$$\deg\left(H(K[V]^G,t)\right) \leq -\dim(V).$$

2.3 An example: $\mathrm{SL}_2(p)$

In this section we consider the action of $G := \mathrm{SL}_2(p)$ on a semisimple KG-module, where $\mathrm{char}(K) = p$. As a Sylow p-subgroup P of G we can take the set of all upper unipotent matrices, and then the normalizer $N = \mathcal{N}_G(P)$ is the set of all upper triangular matrices in $\mathrm{SL}_2(p)$. A complement H of P in N is given by $H = \{\left(\begin{smallmatrix} w & 0 \\ 0 & w^{-1}\end{smallmatrix}\right) \mid w \in \mathbb{F}_p^\times\}$. H acts on the one-dimensional KN-module V_α given by conjugation on P as

$$V_\alpha\colon \begin{pmatrix} w & 0 \\ 0 & w^{-1}\end{pmatrix} \mapsto w^2.$$

Thus α has a square root β in $\mathrm{Irr}(H)$, and we can take $\widetilde{N} = N$ and $V_\beta\colon \left(\begin{smallmatrix} w & 0 \\ 0 & w^{-1}\end{smallmatrix}\right) \mapsto w$.

Let U be the natural, two-dimensional KG-module. The the simple KG-modules are given by the symmetric powers $S^n(U)$ with $0 \leq n < p$ (see Alperin [2, pp. 14–16]). Thus a semisimple KG-module is of the form

$$V = S^{n_1}(U) \oplus \cdots \oplus S^{n_k}(U) \quad (0 \leq n_i < p).$$

It is readily seen that with the notation of Proposition 1.1 we have $S^n(U)_N \cong V_{n+1,\beta^{-n}}$. Thus $V_N \cong V_{n_1+1,\beta^{-n_1}} \oplus \cdots \oplus V_{n_k+1,\beta^{-n_k}}$. From Proposition 2.1(a) we obtain the second part of $T\left(\sigma_t(V)\right)$ as

$$\frac{1}{4p|\widetilde{H}|} \sum_{\tau \in \widetilde{H}} \sum_{\substack{\gamma \in M_{2p},\\ \gamma \neq \pm 1}} \frac{(\gamma-\gamma^{-1})^2}{(1-\beta(\tau)\gamma)(1-\beta(\tau)\gamma^{-1})} \cdot \Phi_{\tau,\gamma}\left(\sigma_t(V)\right) =$$

$$\frac{1}{4p(p-1)} \sum_{\omega \in M_{p-1}} \sum_{\substack{\gamma \in M_{2p},\\ \gamma \neq \pm 1}} \frac{(\gamma-\gamma^{-1})^2}{(1-\omega\gamma)(1-\omega\gamma^{-1})} \prod_{i=1}^{k} \frac{1-(\gamma^{n_i}t)^p}{1-\omega^{-n_i}t^p} \prod_{j=0}^{n_i} (1-\gamma^{n_i-2j}t)^{-1}, \tag{2.7}$$

where as usual $M_n \subset \mathbb{C}$ denotes the set of n-th roots of unity. The summation over the $\sigma \in G_{p'}$ occurring in $T\left(\sigma_t(V)\right)$ can be written as a summation over the p-regular conjugacy classes, which are represented by their eigenvalues in $\mathbb{F}_p$ or $\mathbb{F}_{p^2}$. The result is

$$\frac{1}{|G|} \sum_{\rho \in G_{p'}} \mathrm{ch}_\rho\left(\sigma_t(V)\right) = \frac{1}{2(p-1)} \sum_{\omega \in M_{p-1}} C_\omega + \frac{1}{2(p+1)} \sum_{\omega \in M_{p+1}} C_\omega - \frac{1}{p}(C_1 + C_{-1}) \tag{2.8}$$

with

$$C_\omega := \prod_{i=1}^{k} \prod_{j=0}^{n_i} (1-\omega^{n_i-2j}t)^{-1}$$

(see Srinivasan [16]). By Theorem 2.4, the desired Hilbert series $H(K[V]^G, t)$ is the difference of (2.7) and (2.8). We want to find a way to evaluate the formulas (2.7) and (2.8) without actually performing summations over $(p-1)$-st, $(p+1)$-st and $(2p)$-th roots of unity. The idea is to first perform a partial fraction decomposition on the expressions in the sums as functions in ω. With this, we can decompose the sum over ω into a sum of expressions of the type $A \cdot f_i(n, m, r, a)$ with

$$f_i(n, m, r, a) := \sum_{\omega \in M_n} \frac{\omega^r}{(1-\omega^m a)^i}, \tag{2.9}$$

a and A rational functions in t and possibly γ, $0 \le r < m$, and $n \in \{p-1, p+1\}$. This follows from the special form of the denominators which occur in (2.7) and (2.8). The computation of the $f_i(n, m, r, a)$ is possible by the following proposition.

Proposition 2.6. *Let m, n and r be integers with $m, n > 0$ and $0 \le r < m$, and let a be an indeterminate. Set $d := \gcd(m, n)$, and if $d = 1$, choose $l \in \mathbb{Z}$ such that $nl \equiv -1 \mod m$. Then the following formulas hold for $f_i(n, m, r, a)$ as defined by (2.9).*

$$\begin{aligned} f_{i+1}(n, m, r, a) &= f_i(n, m, r, a) + \frac{a}{i} \cdot \frac{\partial}{\partial a} f_i(n, m, r, a), &(2.10)\\ f_1(n, m, r, a) &= \begin{cases} d \cdot f_1(n/d, m/d, r/d, a) & \text{if } d \text{ divides } r \\ 0 & \text{if } d \nmid r \end{cases}, &(2.11)\\ f_1(n, m, r, a) &= \frac{n \cdot a^{n[(lr-1)/m]+n-r(nl+1)/m}}{1-a^n} \quad \text{if} \quad d = 1, &(2.12) \end{aligned}$$

where the square brackets in (2.12) denote the greatest integer function.

Proof. (2.10) follows from an easy calculation. For $i = 1$, we have

$$f_1(n, m, r, a) = \sum_{k=0}^{\infty} \sum_{\omega \in M_n} a^k \omega^{km+r} = n \cdot \sum_{\substack{k \ge 0, \\ n \mid km+r}} a^k, \tag{2.13}$$

which yields (2.11). If $d = 1$, (2.13) leads to $f_1(n, m, r, a) = na^{k_0}/(1-a^n)$, where k_0 is the smallest non-negative integer with $k_0 m + r \equiv 0 \mod n$. We have to show that

$$k_0 = n \left[\frac{lr-1}{m} \right] + n - r \frac{nl+1}{m} =: k_1.$$

Indeed, k_1 is an integer by the definition of l, and $k_1 m + r \equiv -r(nl+1) + r \equiv 0 \mod n$. An easy calculation shows that $k_1 - n < 0 \le k_1$ is equivalent with

$$-1 - \frac{r}{n} < ((lr-1) \mod m) \le m - 1 - \frac{r}{n}, \tag{2.14}$$

where we write $(x \mod m)$ for the smallest non-negative integer y with $x \equiv y \mod m$. (2.14) is correct for $r = 0$. For $r > 0$, $((lr-1) \mod m) = (lr \mod m) - 1$, and (2.14) is equivalent with $-r < (nlr \mod (nm)) \le nm - r$. But this is true since $nlr \equiv -r \mod m$. □

Notice that the numbers d and l in Proposition 2.6 depend on the congruence class of n modulo m. Using partial fraction decomposition and Proposition 2.6, we can perform the summations over ω. To evaluate (2.7) we can then do a partial fraction decomposition with the resulting rational function in t and γ as a function of γ and then apply Proposition 2.6 again to evaluate the summation over γ. This gives a method which allows us to evaluate $K[V]^G$ for given values of $n_1, \ldots, n_k$, but symbolically for general p. The result is a rational function H_x in the symbols p, t and t^p for each congruence class x of p modulo some integer m. The second author has implemented this method in MAGMA.

Example 2.7. We give the Hilbert series for the invariant ring of $G = \mathrm{SL}_2(p)$ for a few examples of semisimple modules. The calculations were done with the MAGMA program mentioned above.

(a) For $V = U$, the natural module, we obtain

$$H(K[V]^G, t) = \frac{1}{(1 - t^{p+1})(1 - t^{p(p-1)})}.$$

This is the expected answer, since $K[V]^G$ is known to be isomorphic to a polynomial ring, generated by invariants of degrees $p + 1$ and $p(p - 1)$ (see Smith [15, Theorem 8.1.8]).

(b) $V = S^2(U)$ the Hilbert series is

$$H(K[V]^G, t) = \frac{1 + t^{p(p+1)/2}}{(1 - t^2)(1 - t^{p+1})(1 - t^{p(p-1)/2})}.$$

We checked this formula for $p = 3, 5$ and 7 by calculating $K[V]^G$ explicitly and determining the Hilbert series a posteriori.

(c) For $V = U \oplus U$ we obtain

$$H(K[V]^G, t) = \frac{(1 - t^{2p^2}) + pt^{p(p-1)}(1 - t^{2p}) + 2t^{p+1}(1 - t^{(p-1)^2-2})\frac{1-t^{p(p+1)}}{1-t^{p+1}}}{(1 - t^2)(1 - t^{p+1})^2(1 - t^{p(p-1)})^2}.$$

The invariant ring is Cohen-Macaulay by Ellingsrud and Skjelbred [6], since $\dim(V^P) + 2 = \dim(V)$, and by the form of $H(K[V]^G, t)$ it is in fact Gorenstein. There are primary invariants of degrees $p+1, p+1, p(p-1), p(p-1)$, and the degrees of the corresponding secondary invariants can be read off from the Hilbert series. We checked the above formula for $p = 2, 3$ and 5.

(d) For $V = S^3(U)$ we obtain a case distinction according to the congruence class of p modulo 3. For $p \equiv 1 \mod 3$, we have $H(K[V]^G, t) = A/B$ with

$$\begin{aligned} A \quad = \quad & 1 - t^{p-3} + t^{p+1} + t^{3p-3} - t^{3p+1} - t^{4p} - t^{(p^2+p-2)/3} - t^{(p^2+2p-3)/3} - t^{(p^2+2p+3)/3} \\ & + t^{(p^2+4p-11)/3} + t^{(p^2+4p+7)/3} + t^{(p^2+5p+12)/3} - t^{(p^2+7p-2)/3} + t^{(p^2+8p-3)/3} \\ & - t^{(p^2+8p+9)/3} + t^{(p^2+11p)/3} \end{aligned}$$

and

$$B = (1 - t^4)(1 - t^{p-3})(1 - t^{p-1})(1 - t^{p+1})(1 - t^{p+3})(1 - t^{p(p-1)/3}).$$

For $p \equiv -1 \mod 3$, we have $H(K[V]^G, t) = A/B$ with

$$\begin{aligned} A \quad = \quad & 1 - t^{p-3} + t^{p+1} + t^{3p-3} - t^{3p+1} - t^{4p} + t^{(p^2+p+6)/3} - t^{(p^2+3p-4)/3} - t^{(p^2+4p-9)/3} \\ & - 2t^{(p^2+4p-3)/3} + t^{(p^2+4p+9)/3} + t^{(p^2+6p-13)/3} + t^{(p^2+6p+5)/3} + t^{(p^2+7p-12)/3} \\ & + t^{(p^2+7p-6)/3} - t^{(p^2+7p+12)/3} - t^{(p^2+9p-4)/3} - t^{(2p^2+2p-6)/3} + t^{(2p^2+2p+6)/3} \\ & + t^{(2p^2+2p+12)/3} - t^{(2p^2+4p+2)/3} - t^{(2p^2+5p+3)/3} - t^{(2p^2+5p+9)/3} + t^{(2p^2+7p-7)/3} \\ & + t^{(2p^2+7p+11)/3} + t^{(2p^2+8p-6)/3} + t^{(2p^2+8p)/3} - t^{(2p^2+8p+6)/3} - t^{(2p^2+10p+2)/3} \\ & - t^{(3p^2-p+2)/3} - t^{p^2-1} - t^{p^2+1} + t^{(3p^2+2p-7)/3} + t^{(3p^2+2p+11)/3} + t^{p^2+p+4} \\ & - t^{(3p^2+5p+2)/3} + t^{p^2+2p-1} - t^{p^2+2p+3} + t^{p^2+3p} \end{aligned}$$

and

$$B = (1 - t^4)(1 - t^{p-3})(1 - t^{p-1})(1 - t^{p+1})(1 - t^{p+3})(1 - t^{p(p-1)}).$$

In this example $K[V]^G$ is not Cohen-Macaulay (except if $p = 7$; see Shank and Wehlau [14]). We checked the above formulas for $p = 5$ and 7. For $p \leq 19$, we computed all homogeneous invariants of degree up to 24 and checked that the dimensions of the homogeneous subspaces are correctly predicted by the above formulas. ◁

2.4 Cohomology and depth

As the invariant ring $K[V]^G$ is the zeroth cohomology $H^0(G, K[V])$, we may also be interested in higher cohomology modules $H^i(G, K[V])$. These modules were used by Ellingsrud and Skjelbred [6] and by the second author [10, 11] to obtain results about the Cohen-Macaulay property and the depth of modular invariant rings. The $H^i(G, K[V])$ are graded vector spaces (in fact, graded modules over $K[V]^G$), where the grading is given by

$$H^i(G, K[V]) = \bigoplus_{r=0}^{\infty} H^i\left(G, S^r(V)\right).$$

We define the Hilbert series as

$$H\left(H^i(G, K[V]), t\right) := \sum_{r=0}^{\infty} \dim H^i\left(G, S^r(V)\right) \cdot t^r.$$

Again we assume that G is divisible by $p = \operatorname{char}(K)$ but not by p^2, and we retain the notation introduced at the beginning of Section 2. Then by Benson [3, Corollary 3.6.19] we have

$$H^i(G, V) \cong H^i(N, V_N)$$

for V a KG-module and $i > 0$. Therefore $H\left(H^i(G, K[V]), t\right)$ can be computed by a weighted sum over the decomposition series $H_{V_{n,\chi}}(K[V_N], t)$, where the weight is the dimension of $H^i(N, V_{n,\chi})$. For $n = p$, $V_{p,\chi}$ is projective and therefore has no positive cohomology, so we only have to take the "easy" decomposition series for $n < p$ into account. Since $N = P \rtimes H$ and $p \nmid |H|$, it is well known that

$$H^i(N, V_{n,\chi}) \cong H^i(P, V_{n,\chi})^H$$

(see Benson [3, Corollary 3.6.19]). Moreover, since P is cyclic, $H^i(P, V_{n,\chi})$ is isomorphic to the socle or the top of $V_{n,\chi}$ for i even or odd, respectively. The action of H on $H^i(P, V_{n,\chi})$ was explicitly determined by Kemper [11, Proposition 3.2]. From this we derive that

$$\dim\left(H^{2i+\epsilon-1}(N, V_{n,\chi})\right) = \delta_{\chi,\alpha^{i+\epsilon(1-n)}} \quad \text{for} \quad i > 0,\ \epsilon \in \{0, 1\},\ 1 \leq n < p.$$

Proposition 2.8. *For $i > 0$ and $\epsilon \in \{0, 1\}$, define the operator*

$$T_{2i+\epsilon-1} := \frac{1}{4|\widetilde{N}|} \sum_{\tau \in \widetilde{H}} \sum_{\substack{\gamma \in M_{2p},\\ \gamma \neq \pm 1}} \frac{\beta(\tau)^{-2i}(\gamma - \gamma^{-1})^2\left((\beta(\tau)\gamma)^p - \beta(\tau)^{2(1-\epsilon)}\right)}{(1 - \beta(\tau)\gamma)(1 - \beta(\tau)\gamma^{-1})} \cdot \Phi_{\tau,\gamma}$$

on R_{KG}. Then for a KG-module V we have

$$T_{2i+\epsilon-1}(V) = \dim\left(H^{2i+\epsilon-1}(G, V)\right).$$

Proof. By the preceding argument we have

$$\dim\left(H^{2i+\epsilon-1}(G, V)\right) = \dim\left(H^{2i+\epsilon-1}(N, V_N)\right) = \sum_{n=1}^{p-1} T_{n,\alpha^{i+\epsilon(1-n)}}(V)$$

with $T_{n,\chi}$ defined in Proposition 2.2. The (geometric) sum $\sum_{n=1}^{p-1} T_{n,\alpha^{i+\epsilon(1-n)}}$ yields $T_{2i+\epsilon-1}$ as defined in the proposition. $\square$

It is now clear that the Hilbert series $H\left(H^{2i+\epsilon-1}(G, K[V]), t\right)$ can be calculated by applying $T_{2i+\epsilon-1}$ to $\sigma_t(V)$, where the $\Phi_{\tau,\gamma}\left(\sigma_t(V)\right)$ are given by Proposition 2.1. For groups G of order divisible by $p = \mathrm{char}(K)$ but not by p^2, there is a connection between the $H^i(G, K[V])$ and the depth of the invariant ring $K[V]^G$, given by Kemper [11, Theorem 3.1]: We have

$$\mathrm{depth}(K[V]^G) = 1 + \dim(V^P) + \big(\text{the smallest positive } i \text{ with } H^i(G, K[V]) \neq 0\big) \tag{2.15}$$

if the right hand side does not exceed $\dim(V)$. (If the right hand side does exceed $\dim(V)$, then $\mathrm{depth}(K[V]^G) = \dim(V)$.) Since we can compute the Hilbert series of the cohomologies, we can also decide which cohomologies are non-zero. Thus we have a method to calculate the depth of $K[V]^G$ without touching a single invariant. This method has roughly the same computational complexity as evaluating Molien's formula.

We apply this method to prove a conjecture of the second author [11, Conjecture 5.2(a)].

Theorem 2.9. *Let U be the natural two-dimensional module of $G := \mathrm{GL}_2(p)$ over a field K of characteristic p, and let $V := S^3(U)$ be the third symmetric power. Then $K[V]^G$ is Cohen-Macaulay.*

Remark. If $V = S^n(U)$ with $n < 3$, then $\dim(V) \leq 3$ and so $K[V]^G$ is Cohen-Macaulay by Ellingsrud and Skjelbred [6]. ◁

Proof of Theorem 2.9. We first check the Cohen-Macaulayness of $K[V]^G$ for $p = 2$ and 3 by explicitly computing the invariant ring using MAGMA (see Kemper and Steel [12]). Indeed, for $p = 2$ the invariant ring is a hypersurface, and for $p = 3$ it is a Gorenstein ring. Now we assume that $p > 3$.

By Equation (2.15), we have to prove that $H^1(G, K[V]) = 0$, since $\dim(V) = 4$ and $\dim(V^*)^P = 1$. For P we choose the subgroup of upper unipotent matrices, and then $N = \mathcal{N}_G(P)$ is the group of upper triangular matrices. The diagonal matrices form a complement H of P in N. The representation V_α of H is given by

$$\begin{pmatrix} w & 0 \\ 0 & 1 \end{pmatrix} \mapsto w, \quad \begin{pmatrix} w & 0 \\ 0 & w \end{pmatrix} \mapsto 1.$$

The restriction of $V = S^3(U)$ to N is the KN-module $V_{4,\chi}$, where V_χ is the one-dimensional module given by

$$\begin{pmatrix} w & 0 \\ 0 & 1 \end{pmatrix} \mapsto 1, \quad \begin{pmatrix} w & 0 \\ 0 & w \end{pmatrix} \mapsto w^3.$$

Using Propositions 2.8 and 2.1, we obtain

$$H\left(H^1(G, K[V]), t\right) = T_1\left(\sigma_t(V)\right) =$$
$$\sum_{\zeta \in M_{2(p-1)}} \sum_{\omega \in M_{p-1}} \sum_{\substack{\gamma \in M_{2p}, \\ \gamma \neq \pm 1}} \frac{(\gamma - \gamma^{-1})^2 (\zeta^{p-2}\gamma^p - 1)(1 - (\zeta\omega\gamma)^{3p} t^p)}{8p(p-1)^2 (1 - \zeta\gamma)(1 - \zeta\gamma^{-1})(1 - (\omega^3 t)^p)} \cdot \prod_{j=0}^{3} (1 - (\zeta\omega)^3 \gamma^{3-2j} t)^{-1},$$

where as usual $M_n \subset \mathbb{C}$ is the set of n-th roots of unity. The calculation of this sum can be performed for general p by using the ideas of Section 2.3. We get different cases for every congruence class for p modulo 36. In each case the sum turns out to be zero. The computations were done with MAGMA and took about two hours of computation time. □

We finish by giving a brief summary of what is known to date about invariant rings of groups whose order is not divisible by p^2. Let G be a finite group, K a field of characteristic p, and V an n-dimensional KG-module. Assume that $|G|$ is not divisible by p^2. Then:

- $K[V]^G$ is generated by invariants of degree at most $\max\{n(|G| - 1), |G|\}$ (Hughes and Kemper [8, Theorem 2.17]).
- For $i > 0$, the cohomology module $H^i(G, K[V])$ is Cohen-Macaulay as a module over $K[V]^G$ (Kemper [11, Theorem 2.12(c)]).

- $\deg_t\left(H\left(K[V]^G, t\right)\right) \leq -n$ (Corollary 2.5 in this article).
- We have a method to calculate the Hilbert series and the depth of $K[V]^G$ without computing any invariants (this article).

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