The Depth of Invariant Rings and Cohomology

Gregor Kemper
with an appendix by Kay Magaard

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Abstract

Let G be a finite group acting linearly on a vector space V over a field K of positive characteristic p, and let $P \leq G$ be a Sylow p-subgroup. Ellingsrud and Skjelbred [20] proved the lower bound

$$\operatorname{depth}(K[V]^G) \geq \min\{\dim(V^P) + 2, \dim(V)\}$$

for the depth of the invariant ring, with equality if G is a cyclic p-group. Let us call the pair (G, V) *flat* if equality holds in the above. In this paper we use cohomological methods to obtain information about the depth of invariant rings and in particular to study the question of flatness. For G of order not divisible by p^2 it ensues that (G, V) is flat if and only if $H^1(G, K[V]) \neq 0$ or $\dim(V^P) + 2 \geq \dim(V)$. We obtain a formula for the depth of the invariant ring in the case that G permutes a basis of V and has order not divisible by p^2. In this situation (G, V) is usually not flat. Moreover, we introduce the notion of visible flatness of pairs (G, V) and prove that this implies flatness. For example, the groups $\mathrm{SL}_2(q)$, $\mathrm{SO}_3(q)$, $\mathrm{SU}_3(q)$, $\mathrm{Sz}(q)$, and $\mathrm{R}(q)$ with many interesting representations in defining characteristic are visibly flat. In particular, if $G = \mathrm{SL}_2(q)$ and V is the space of binary forms of degree n or a direct sum of such spaces, then (G, V) is flat for all $q = p^r$ with the exception of a finite number of primes p.

Along the way, we obtain results about the Buchsbaum property of invariant rings, and about the depth of the cohomology modules $H^i(G, K[V])$. We also determine the support of the positive cohomology $H^+(G, K[V])$ as a module over $K[V]^G$. In the appendix, the visibly flat pairs (G, V) of a group with BN-pair and an irreducible representation are classified.

Contents

Introduction

We consider invariants of a finite group G acting on a polynomial ring $R = K[x_1, \dots, x_n]$ by linear transformations of the indeterminates. The set R^G of invariants forms a (graded) algebra over the ground field K, and we are interested in the structure of this algebra. By Noether [45] the invariant ring is finitely generated as a K-algebra. More precisely, by Noether normalization there exist homogeneous invariants $f_1, \dots, f_n \in R^G$ such that R^G is finitely generated as a module over the subalgebra $A := K[f_1, \dots, f_n]$ generated by the f_i (see Benson [6, Theorem 2.2.7]). Since the number of f_i equals the number of indeterminates, $f_1, \dots, f_n$ are algebraically independent over K, so the subalgebra A is isomorphic to a polynomial algebra. An important aspect of the complexity of the invariant ring is therefore captured by looking at its complexity as a module over A. The simplest case is that R^G is a free A-module. Then R^G is called **Cohen-Macaulay**. By the theorem of Hochster and Eagon [28], R^G is always Cohen-Macaulay if the characteristic of K does not divide the group order $|G|$ (which includes the case of characteristic 0). Therefore it remains to consider the **modular case**, where the characteristic divides $|G|$. Here indeed we have many examples where the invariant ring is not Cohen-Macaulay (see Kemper [37] and the references there). Consider a minimal free resolution

$$0 \longrightarrow F_m \longrightarrow F_{m-1} \longrightarrow \cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow R^G \longrightarrow 0$$

of R^Gas an A-module, which by Hilbert's syzygy theorem has finite length $m \le n$. The image of F_1 in F_0 gives the A-linear relations between the generators of R^G as an A-module ("syzygies of the first kind"), whereas the image of F_2 gives the relations ("syzygies of the second kind") between these first relations, and so on. The length m is independent of the choice of the resolution, and indeed even of the choice of the $f_1, \dots, f_n$ (see Benson [6, Section 4.4]). It is called the **homological dimension** of R^G and provides the desired measure of the complexity. We write $m = \mathrm{hdim}(R^G)$. The homological dimension of R^G is linked to the depth by the famous Auslander-Buchsbaum formula (see Benson [6, Theorem 4.4.4]), which says that

$$\mathrm{hdim}(R^G) = n - \mathrm{depth}(R^G).$$

We will recall the definition of depth below, but for the sake of this introduction one might as well take the above formula as a definition. In particular, R^G is Cohen-Macaulay if and only if $\mathrm{depth}(R^G) = n$.

Previous work. The question of the depth of modular invariant rings has enjoyed considerable interest recently (see Bourguiba and Zarati [8], Smith [52], Campbell et al. [14], Shank and Wehlau [49]). However, the story of results (known to the author) which study the depth of modular invariant rings for certain classes of groups is rather short. In 1980, Ellingsrud and Skjelbred [20] proved in a ground-breaking paper that if G is a cyclic p-group, where p is the characteristic of the ground field K, then

$$\operatorname{depth}(R^G) = \min\left\{\dim(V^G) + 2, n\right\}. \tag{0.1}$$

Here V is the vector space on which G acts, V^G is the fixed space under the G-action, and basis vectors $x_1, \ldots, x_n$ of the dual V^* are taken as the indeterminates of R. Ellingsrud and Skjelbred also proved that for any finite group G we have the inequality

$$\operatorname{depth}(R^G) \geq \min\left\{\dim(V^P) + 2, n\right\}, \tag{0.2}$$

where $P \leq G$ is a Sylow p-subgroup. We say that the pair (G, V) is **flat** if equality holds in (0.2). Since a p-group always fixes non-zero vectors, it follows from (0.2) that $\operatorname{depth}(R^G) \geq \min\{3, n\}$, so in particular, R^G is Cohen-Macaulay if $n \leq 3$ (see Smith [51]). The result of Ellingsrud and Skjelbred on cyclic p-groups was recently generalized by Campbell et al. [14], who proved that if G acts a shallow group on V (see Remark 2.8(c) in this paper), then (G, V) is flat. Typical examples of shallow groups are abelian groups with a cyclic Sylow p-subgroup P. Unlike Ellingsrud and Skjelbred's proof, the proof in [14] uses elementary methods. The most recent contribution comes from Shank and Wehlau [49], who studied the depth of invariants of $\mathrm{SL}_2(p)$ acting on symmetric powers of the natural two-dimensional representation. The interest in the action of SL_2 on binary forms of certain degrees (the classical term for symmetric powers of the natural representation) stems from the fact that they have been studied (in the non-modular setting) very intensively in classical invariant theory (see Hilbert [27]). Shank and Wehlau proved that if R^G is the invariant ring of $G = \mathrm{SL}_2(p)$ acting on the m-th symmetric power V of the two-dimensional representation such that $m < p$ and $\gcd(m, p-1) \leq 2$, then (G, V) is flat, so for $m > 1$ we have

$$\operatorname{depth}(R^G) = 3. \tag{0.3}$$

In addition, Shank and Wehlau obtained complete information about the depth for $m = 3$ and 4. We observe that flatness emerges as a common pattern from the results quoted above. However, the symmetric group S_p on p symbols has an invariant ring which is isomorphic to a polynomial ring even in characteristic p. Hence the depth is p, whereas the dimension of the fixed space of P is only 1. This suggest that in addition to the dimension of V^P there should be other, hitherto undiscovered parameters which contribute to the depth of the invariants.

Methods and results. This paper can be seen as an attempt to find such parameters, or, more precisely, to find them in the cohomology $H^*(G, R)$ of G with values in the polynomial ring R. In fact, this cohomology was used by Ellingsrud and Skjelbred [20] and Kemper [37], but it is not apparent from these papers that it can play a major role in the determination of the depth for non-cyclic groups. In [37], the following result was used: If r is the smallest positive integer such that $H^r(G, R) \neq 0$ and if $0 \neq \alpha \in H^r(G, R)$, then

$$\operatorname{depth}\left(\operatorname{Ann}_{R^G}(\alpha), R^G\right) = \min\left\{r + 1, \operatorname{ht}\left(\operatorname{Ann}_{R^G}(\alpha)\right)\right\},$$

where $\operatorname{Ann}_{R^G}(\alpha)$ is the ideal in R^G of all invariants f with $f\alpha = 0$. This prompts the study of $H^+(G, R) := \bigoplus_{i>0} H^i(G, R)$ as an R^G-module, and in particular of the varieties $\mathcal{V}_\alpha$ given by $\operatorname{Ann}_{R^G}(\alpha)$ for $\alpha \in H^+(G, R)$. In [37], the author was only able to determine $\mathcal{V}_\alpha$ in the case where α comes from the embedding $H^1(G, K) \hookrightarrow H^1(G, R)$. In Section 2 of this article further methods are developed, which lead to an upper bound for the variety $\mathcal{V}_\alpha$. The main ingredient is re-discovery of the result that every invariant in the image of the transfer map lies in a projective submodule of R (see Proposition 2.2). Using the Evens norm, we are able to determine the support of $H^+(G, R)$ (which is the union of all $\mathcal{V}_\alpha$) exactly. The upper bound for $\mathcal{V}_\alpha$ yields the R^G-depth of certain ideals. This information is complemented by results from Section 1 about the extendability of regular sequences. More precisely, we prove that if $U \leq V$ is a KG-submodule on which G acts non-modularly, then

$$\operatorname{depth}(R^G) = \operatorname{depth}\left(\mathcal{I}_{R^G}(U), R^G\right) + \dim_K(U), \tag{0.4}$$

where $\mathcal{I}_{R^G}(U)$ is the ideal of invariants vanishing on U (see Theorem 1.5). Putting the two strands together, we can determine the depth if we find $0 \neq \alpha \in H^r(G,R)$ (r being the smallest positive integer with $H^r(G,R) \neq 0$) such that

$$\mathcal{V}_\alpha \subseteq U$$

for a KG-submodule U with non-modular G-action. A careful analysis of this condition leads to Theorem 2.6, which gives the depth under a certain hypothesis on the cohomology. This result contains the result by Campbell et al. [14] on shallow groups and therefore also the result of Ellingsrud and Skjelbred [20] (see Equation (0.1)) as special cases. Theorem 2.6 exemplifies how the dimension of the fixed space under a Sylow p-subgroup (with $p = \operatorname{char}(K)$) and the smallest positive r with $H^r(G,R) \neq 0$ contribute equal shares to the depth.

As a by-product of Theorem 1.5 we get new results concerning the relation between the Buchsbaum property and the Cohen-Macaulay property. In the more general setting of graded algebras, the Buchsbaum property is weaker than the Cohen-Macaulay property, but Nakajima [43] and Campbell et al. [14] independently proved that in the case of a p-group the invariant ring is Cohen-Macaulay if and only if it is Buchsbaum. Using Equation (0.4), we can generalize this equivalence result to a larger class of groups (see Theorem 1.7), which for example includes all groups whose order is not divisible by p^2. It is still an open question whether the two properties are equivalent for any invariant ring of a finite group acting via a linear representation. As a further spin-off from the methods developed in Sections 1 and 2, we obtain results on the depth of $H^+(G,R)$ as a module over R^G (see Section 2.2). For example, if p^2 does not divide the order $|G|$, then $H^+(G,R)$ is Cohen-Macaulay (see Example 2.14). This generalizes a result by Ellingsrud and Skjelbred [20], who proved the above statement for G the cyclic group of order p.

A fruitful generalization (Corollary 2.9) of Theorem 2.6 comes from the observation made by the author [37] that $\operatorname{depth}(R^G) = \operatorname{depth}(R^H)$ if $H \leq G$ is a strongly p-embedded subgroup (see page 12 for a definition). It follows, for example, that we can determine the depth for a group G of order divisible by p but not by p^2 if we know the smallest positive r such that $H^r(G,R) \neq 0$ (see Theorem 3.1). This r can be determined quite easily under the additional hypothesis that G acts as a permutation group. This is carried out in Section 3, where we give a formula for the depth of the invariant ring of a permutation group whose order is not divisible by p^2 (see Theorem 3.3). In particular, this yields a cohomological explanation for the “anomaly” of the large depth for $G = S_p$ mentioned above. In fact, the vector invariants of two copies of the natural module of S_p are also Cohen-Macaulay, while for three copies we have the depth $2p+1$ due to the non-vanishing of the $(2p-3)$rd cohomology (see Corollary 3.5). We also explore applications of this formula to the regular representation and the “three copies conjecture” (see Corollaries 3.4 and 3.6).

A great part of the literature on cohomology deals with cohomology $H^*(G,K)$ with values in the ground field (or in $\mathbb{Z}$). Looking at $H^*(G,R)$ poses additional difficulties, but also extends the possibilities of finding non-zero elements. For example, $H^1(G,K) = 0$ unless G has a normal subgroup of index p, but $H^1(G,R) \neq 0$ if p divides $|G|$ and V is the regular KG-module (see Kemper [37, Lemma 2.6]). In Section 2.3 we give conditions on a subgroup $N \leq G$, under which non-zero elements from $H^*(N,R)$ can be twisted in such a way as to give non-zero elements of the same degree in $H^*(G,R)$ (see Theorem 2.15). Applying this to the first cohomology with N being a Sylow p-subgroup leads to the result that visible flatness of (G,V) (as defined in Definition 4.1) implies flatness. It turns out that subgroups of groups with a BN-pair of rank one with various representations tend to be visibly flat. In the appendix, Kay Magaard determines the visibly flat pairs (G,V) of a group with BN-pair and an irreducible KG-module. For instance, the groups $\mathrm{SL}_2(q)$ acting on binary forms of degree m such that $\gcd(m,q-1) \leq 2$ are visibly flat. Hence the result of Shank and Wehlau [49] (see Equation (0.3)) appears as a special case of Theorem 4.2. Moreover, (subgroups of) $\mathrm{SO}_3(q)$, $\mathrm{SU}_3(q)$, $\mathrm{Sz}(q)$, $\mathrm{R}(q)$, and a few sporadic simple groups with various representations are visibly flat, so we can determine the depth of the invariant rings.

Section 5 is devoted to the study of the depth of invariants of $\mathrm{SL}_2(q)$ and $\mathrm{GL}_2(q)$ acting on binary forms of certain degrees, where in general we do not have visible flatness. Again we develop methods to evaluate $H^+(G,R)$ in this case. Using these, we can compute tables giving the depths of invariants of $\mathrm{SL}_2(p)$ and $\mathrm{GL}_2(p)$ acting on binary forms of degree m for wide ranges of p and m (see Tables 5.1 and 5.2). Table 5.1 includes the results of Shank and Wehlau [49] for $m = 3$ and 4. Using methods from classical invariant theory (see Hilbert [27]), we prove that for any given degree m there exists a bound N such that for $q = p^r$ with $p \geq N$ the invariants of $\mathrm{SL}_2(q)$ acting on binary forms of degree m have depth 3, so we have flatness. This appears in a more general form in Theorem 5.4. Various questions raised by Shank and Wehlau [49] are answered in Section 5.

Many of the results of this paper carry over to the more general setting of equivariants or relative invariants. More precisely, we take a further representation W of G and consider G-equivariant polynomial mappings from V into W. These mappings form a module M over the invariant ring R^G, so we can ask for the depth of M. If W is one-dimensional, M can be interpreted as the module of relative invariants of G with respect to a linear character.

A remark on the depth of localizations. Apart from considering the "global" depth of R^G one can also ask about the local depth, i.e., the depth of localizations $(R^G)_y$ for $y \in \mathrm{Spec}(R^G)$. But this question can be reduced to the computation of the global depth for subgroups by a result from Kemper [36]. Indeed, for a prime ideal $x \in \mathrm{Spec}(R)$ define

$$G_x := \{\sigma \in G \mid (\sigma - 1)R \subseteq x\} \leq G.$$

If x comes from a point in V, then G_x is precisely the point-stabilizer. Then with the morphism $\pi\colon \mathrm{Spec}(R) \to \mathrm{Spec}(R^G)$ induced from the embedding $R^G \subseteq R$ we have

$$\mathrm{depth}\left((R^G)_{\pi(x)}\right) = \mathrm{depth}(R^{G_x}) - \dim(R/x) \qquad (0.5)$$

(see [36, Satz 5.12]). This is deduced from Luna's Slice Theorem. Usually one is interested in zero-dimensional points, i.e., $\dim(R/x) = 0$. By Equation (0.5) the local depth can be determined for any $y \in \mathrm{Spec}(R^G)$ if the global depth is known for every subgroup $H \leq G$. So let us consider our main results. If (G,V) is visibly flat (defined in Definition 4.1), then we see from Proposition 4.3(c) that in many cases (H,V) is visibly flat for every subgroup $H \leq G$ of order divisible by $p = \mathrm{char}(K)$. In particular, this is true for subgroups of $\mathrm{SL}_2(q)$, $\mathrm{SO}_3(q)$, and $\mathrm{SU}_3(q)$ (for the latter, see in the proof of Theorem 4.10). On the other hand, for subgroups H of order not divisible by p we know from the theorem of Hochster and Eagon that $\mathrm{depth}(R^H) = n$. Hence for many significant examples where (G,V) is visibly flat we can determine the local depth at every point. Moreover, if G is a permutation group with $p^2 \nmid |G|$, then this is also true for every subgroup. So here, too, we can determine the local depth at any point by applying Theorem 3.3. Finally, Theorem 5.4 on the generic depth of $\mathrm{SL}_2(q)$-invariants passes down to modular subgroups, so here, too, we can determine the local depth at all points.

Some notation. The following notations will be used throughout the article.

- An associative ring R is called **graded** if $R = \bigoplus_{d\in\mathbb{Z}} R_d$ with $R_i \cdot R_j \subseteq R_{i+j}$. If $R_d = 0$ for $d < 0$, R is said to be **non-negatively graded**. Then we write $R_+ := \bigoplus_{d>0} R_d$ for the ideal generated by all homogeneous elements of positive degree. A module M over a graded algebra is called graded if $M = \bigoplus_{d\in\mathbb{Z}} M_d$ such that $A_i M_j \subseteq M_{i+j}$.

- Let M be a module over a commutative, associative ring R with unity. Then for $m \in M$ we write

  $$\mathrm{Ann}_R(m) := \{f \in R \mid fm = 0\},$$

  and $\mathrm{Ann}_R(M) := \bigcap_{m\in M} \mathrm{Ann}_R(m)$.

In the following R is a commutative, associative ring R with unity, and M is an R-module.

- The **support** of M (written as $\operatorname{Supp}(M)$) is the set of all prime ideals in R containing $\operatorname{Ann}_R(M)$.
- We write $\dim(R)$ for the Krull dimension of R, and $\dim(M) := \dim(R/\operatorname{Ann}_R(M))$. It is convenient to set $\dim(0) = -1$. To avoid confusion, we write $\dim_K(V)$ for the dimension of a vector space V over a field K.
- Let $f_1, \ldots, f_k \in R$. Then

$$(f_1, \ldots, f_k)M := \{f_1 m_1 + \cdots + f_k m_k \mid m_1, \ldots, m_k \in M\}.$$

  This is a submodule of M. In particular, $(f_1, \ldots, f_k)R$ is the ideal generated by $f_1, \ldots, f_k$.
- A sequence $f_1, \ldots, f_k \in R$ is called a **partial system of parameters** if $(f_1, \ldots, f_k)R \neq R$ and $\operatorname{ht}((f_1, \ldots, f_k)R) = k$. Notice that if R is a domain which is finitely generated as an algebra over a field, then $\operatorname{ht}(I) = \dim(R) - \dim(R/I)$ for any ideal $I \subseteq R$ (see Eisenbud [19, Corollary 13.4]). A partial system of parameters of length equal to $\dim(R)$ is called a **system of parameters**. If R is graded, then a (partial) system of parameters is called **homogeneous** if every f_i lies in some R_{d_i}. In finitely generated (non-negatively graded) algebras over a field, (homogeneous) systems of parameters exist by the Noether Normalization Theorem (see Eisenbud [19, Theorem 13.3]).
- A sequence $f_1, \ldots, f_k \in R$ is called M**-regular** if $(f_1, \ldots, f_k)M \neq M$ and multiplication by f_i is injective on the quotient $M/(f_1, \ldots, f_{i-1})M$ for all i.
- For an ideal $I \subseteq R$ we write

$$\operatorname{depth}(I, M) = \sup\{k \mid \text{there exists an } M\text{-regular sequence of length } k \text{ in } I\}.$$

  It is convenient to set $\operatorname{depth}(I, 0) := -1$. If R is Noetherian, M is finitely generated, and $\operatorname{Ann}_R(M) \subseteq \sqrt{I} \subsetneqq R$, then $\operatorname{depth}(I, M)$ is equal to the length of *any* maximal M-regular sequence in I (see Bruns and Herzog [9, Theorem 1.2.5]).
- If R is non-negatively graded and M a graded R-module, then

$$\operatorname{depth}(M) := \operatorname{depth}(R_+, M).$$

  If R is a local ring with maximal ideal P then we write $\operatorname{depth}(M) := \operatorname{depth}(P, M)$.
- For a group G and a field K, the **group ring** is denoted by KG.
- For V a KG-module and $\sigma \in G$, we write

$$V^\sigma := \{v \in V \mid \sigma(v) = v\}.$$

  Moreover, the **fixed space** of G is $V^G := \cap_{\sigma \in G} V^\sigma$.
- If V is a KG-module, we write V^* for the dual module, which is endowed with a G-action by $\sigma(f) = f \circ \sigma^{-1}$. We write $S^m(V)$ for the m-th symmetric power of V, which again is a KG-module. $S(V) := \bigoplus_{d=0}^\infty S^d(V)$ is the **symmetric algebra**, which is a non-negatively graded K-algebra. Notice that the invariant ring $S(V)^G$ is also graded.
- We will often use the fact that for an action of a finite group G on R we have that R is integral over R^G. Indeed, $\prod_{\sigma \in G}(X - \sigma(f))$ provides an integral equation for $f \in R$. In particular, if V is a finite-dimensional KG-module and G is finite, then it follows that

$$\dim\left(S(V^*)^G\right) = \dim_K(V) \tag{0.6}$$

  (see Eisenbud [19, Proposition 9.2]).

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1 Extending regular sequences

In this section we will develop techniques to extend regular sequences which lie in some ideal of the invariant ring.

1.1 Non-modular submodules

The first lemma provides the principle that we will use to extend regular sequences. We consider the following situation: Let R be an algebra (associative, commutative and with unity) over a field K and G a group acting on R by K-automorphisms. Write $E := \mathrm{End}_{KG}(R)$ for the algebra of KG-endomorphisms of R (which need not respect the multiplication in R). For an invariant $f \in R^G$, multiplication with f yields an endomorphism $\mu_f \in E$, so we have a homomorphism $R^G \to E$ of algebras. We will be looking at (left) modules over E. Every such module becomes an R^G-module by the above map. A prominent example of an E-module is the invariant ring R^G itself (since the action of a KG-endomorphism sends R^G to itself). More generally, if W is a KG-module, then an algebra-homomorphism from E into $\mathrm{End}_{KG}(W \otimes_K R)$ is given by sending φ to $\mathrm{id}_W \otimes \varphi$. This makes $W \otimes_K R$ into an E-module. Since the elements of E act on $W \otimes_K R$ by KG-endomorphisms, the cohomology $H^*(G, W \otimes_K R)$ also becomes an E-module. The invariants $(W \otimes_K R)^G$ occur as $H^0(G, W \otimes_K R)$. If W is one-dimensional, $(W \otimes_K R)^G$ can be interpreted as the module of relative invariants in R with respect to a linear character of G.

If R is a graded algebra and the action of G respects the grading, then E is also graded by calling an endomorphism φ homogeneous of degree d if $\varphi(R_i) \subseteq R_{i+d}$ for all i. If R is graded, then in the above example $W \otimes_K R$ becomes a graded E-module by setting $(W \otimes_K R)_d := W \otimes_K R_d$. Therefore $H^*(G, W \otimes_K R) = \bigoplus_{d=0}^{\infty} H^*(G, W \otimes_K R_d)$ also is a graded E-module.

Lemma 1.1. *In the above situation, let M be an E-module and let $f_1, \ldots, f_k \in R^G$ be an M-regular sequence. Furthermore, let $g \in R^G$ be an invariant with $(f_1, \ldots, f_k, g)M \neq M$, and assume that μ_g has a left inverse in $\mathrm{End}_{AG}(R) \subseteq E$, where AG is the group ring over the subalgebra $A := K[f_1, \ldots, f_k] \subseteq R$ generated by the f_i. In other words, assume that there exists $\varphi \in \mathrm{End}_{AG}(R)$ such that*

$$\varphi(gf) = f \quad \text{for all} \quad f \in R.$$

Then $f_1, \ldots, f_k, g$ is also an M-regular sequence.

Proof. We have to show that multiplication with g is injective on $\overline{M} := M/(f_1, \ldots, f_k)M$. For any $\psi \in \mathrm{End}_{AG}(R)$ we have $\psi\mu_{f_i} = \mu_{f_i}\psi$, and thus for $m \in M$ we obtain

$$\psi(f_i m) = \psi(\mu_{f_i} m) = (\psi\mu_{f_i})m = \mu_{f_i}(\psi m) = f_i \psi m \in (f_1, \ldots, f_k)M.$$

Therefore $\psi(f_1, \ldots, f_k)M \subseteq (f_1, \ldots, f_k)M$, so $\overline{M}$ becomes an $\mathrm{End}_{AG}(R)$-module. Hence multiplication with g has a left inverse as an endomorphism of $\overline{M}$ and is therefore injective. □

It is far from clear how and under what conditions the multiplication with g has a left inverse as required in Lemma 1.1. As we will show, in the situation of the following proposition, such inverses can in fact be constructed under some conditions.

Proposition 1.2. *Let G be a finite group, K a field of characteristic p, and V a finite-dimensional KG-module. Furthermore, let $W \le V$ be a KG-submodule such that the kernel $N \trianglelefteq G$ of the G-action on the quotient V/W has index not divisible by p, i.e., $p \nmid [G : N]$. Then with $R := S(V)$, there exist homogeneous $g_1, \ldots, g_m \in R^G$ with the following two properties:*

(a) The images of $g_1, \ldots, g_m$ form a homogeneous system of parameters for the quotient ring $R^G/(W)R \cap R^G$. Here $(W)R \cap R^G$ denotes the intersection of the R-ideal generated by W with R^G.

(b) For each i, multiplication with g_i has a left inverse in $\operatorname{End}_{A_iG}(R)$, where A_i is the subalgebra

$$A_i := S(W)^G[g_1, \ldots, g_{i-1}, g_{i+1}, \ldots, g_m] \subseteq R^G$$

generated by $S(W)^G$ and all g_j with $j \neq i$.

In fact, any $g_1, \ldots, g_m$ having Property (a) also have Property (b), and $m = \operatorname{codim}_V(W)$.

Proof. By the Noether Normalization Theorem, homogeneous $g_1, \ldots, g_m \in R^G$ satisfying Property (a) exist. It follows that the ideal $I \subset R^G$ generated by $(W)R \cap R^G$ and the g_i has Krull dimension zero. Since G is finite, it follows from Eisenbud [19, Proposition 9.2] that every proper ideal in R containing I is also zero-dimensional. In particular, this is true for the ideal $(y_1, \ldots, y_r, g_1, \ldots, g_m)R$, where $y_1, \ldots, y_r$ is a K-basis of W. The ideal $(W)R \subset R$ has height r, and again by Eisenbud [19, Proposition 9.2] the same is true for the ideal $(W)R \cap R^G \subset R^G$. Therefore

$$m = \dim(R^G) - r = \dim(R) - r = \dim_K(V) - \dim_K(W) = \operatorname{codim}_V(W).$$

It follows that $y_1, \ldots, y_r, g_1, \ldots, g_m$ form a homogeneous system of parameters for R. Being a polynomial algebra, R is Cohen-Macaulay and therefore free as a module over the subalgebra $A := K[y_1, \ldots, y_r, g_1, \ldots, g_m]$ (see Bruns and Herzog [9, Theorem 1.3.3 and Proposition 1.5.15]). By the graded version of Nakayama's lemma (see, for example, Kemper [35, Lemma 2.1]), a set $B \subseteq R$ is a basis of R over A if and only if the image of B is a K-basis of $R/(y_1, \ldots, y_r, g_1, \ldots, g_m)R$. Choose $x_1, \ldots, x_m$ such that $y_1, \ldots, y_r, x_1, \ldots, x_m$ form a basis of V. By Gröbner basis theory we can choose a K-basis B of $R/(y_1, \ldots, y_r, g_1, \ldots, g_m)R$ consisting of precisely those monomials in the x_i and y_j which do not lie in the lead term ideal of $(y_1, \ldots, y_r, g_1, \ldots, g_m)R$, where an arbitrary term order is chosen (see Becker and Weispfenning [3, Proposition 6.52]). By the above remark, B will be a basis of R over A. Since any monomial involving a variable y_i lies in the lead term ideal, the properties of B can be summarized as follows.

(i) B is a set of monomials in the variables $x_1, \ldots, x_m$;

(ii) If $b \in B$ and b' is a monomial dividing b, then $b' \in B$;

(iii) B is a basis of R as a free module over $A = K[y_1, \ldots, y_r, g_1, \ldots, g_m]$.

Let M be the set of all monomials in the variables $y_1, \ldots, y_r$. Since $y_1, \ldots, y_r, g_1, \ldots, g_m$ are algebraically independent over K, M is a basis of A as a free module over $A' := K[g_1, \ldots, g_m]$. Therefore the set

$$B \cdot M := \{bm \mid b \in B, m \in M\}$$

is a basis of R as a free A'-module.

On the level of the subalgebra A', left inverses of the μ_{g_i} exist. Indeed, $g_1, \ldots, g_m$ are algebraically independent, hence for $a \in A'$ there exist unique elements $\widetilde{a} \in A'$ and $\hat{a} \in A'_i := K[g_1, \ldots, g_{i-1}, g_{i+1}, \ldots, g_m]$ such that

$$a = g_i \widetilde{a} + \hat{a}.$$

We define ψ_i: $A' \to A'$ by setting $\psi_i(a) = \widetilde{a}$. Then for $i = 1, \ldots, m$ we have:

$$\psi_i \text{ is a homomorphism of } A_i'\text{-modules with } \psi_i(g_i a) = a \text{ for all } a \in A'. \tag{1.1}$$

Now we extend the ψ_i to R as follows: An $f \in R$ has a unique representation

$$f = \sum_{j=1}^{r_f} a_j \cdot b_j m_j$$

with $r_f \in \mathbb{N}$, $a_j \in A'$, and $b_j m_j \in B \cdot M$, where the $b_j m_j$ are distinct for different values of j. Hence we can set

$$\psi_i(f) := \sum_{j=1}^{r_f} \psi_i(a_j) \cdot b_j m_j \in R.$$

We claim that ψ_i: $R \to R$ is a homomorphism of modules over $A_i'[y_1, \ldots, y_r]$ (and therefore also over A_i) which commutes with the action of the subgroup $N \leq G$, such that $\psi_i \circ \mu_{g_i} = \mathrm{id}_R$. It is straight forward to check that the ψ_i are K-linear and commute with the multiplication by a y_k and a g_k with $k \neq i$, and that $\psi_i(g_i f) = f$ for every $f \in R$. Now take a $\sigma \in N$. Since $\sigma(x_j) \in x_j + W$, it follows that for b a monomial in the x_j, $\sigma(b)$ is a K-linear combination of products of the form $b'm$ with b' a monomial in the x_j dividing b and m a monomial in the y_j. By property (ii) above, $b'm \in B \cdot M$ if $b \in M$. Moreover $\sigma(m)$ is a K-linear combination of monomials from M. Therefore,

$$\sigma(b_j m_j) = \sum_{k=1}^{l_j} \alpha_{j,k} \cdot b_{j,k} m_{j,k}$$

with $\alpha_{j,k} \in K$, $b_{j,k} \in B$, and $m_{j,k} \in M$. Since $A' \subseteq R^G$, we have

$$\psi_i(\sigma(f)) = \psi_i\left(\sum_{j=1}^{r_f} a_j \sum_{k=1}^{l_j} \alpha_{j,k} \cdot b_{j,k} m_{j,k}\right) = \sum_{j=1}^{r_f} \sum_{k=1}^{l_j} \alpha_{j,k} \psi_i(a_j) \cdot b_{j,k} m_{j,k}.$$

On the other hand,

$$\sigma(\psi_i(f)) = \sum_{j=1}^{r_f} \psi_i(a_j) \cdot \sigma(b_j m_j) = \sum_{j=1}^{r_f} \psi_i(a_j) \cdot \sum_{k=1}^{l_j} \alpha_{j,k} b_{j,k} m_{j,k},$$

so we have $\psi_i(\sigma(f)) = \sigma(\psi_i(f))$ as desired.

It is easy to check that for $\sigma \in G$, the map $\sigma \circ \psi_i \circ \sigma^{-1}$: $R \to R$ also has the properties that we proved for ψ_i. Now we use the hypothesis on the index $[G : N]$ and form

$$\varphi_i := \frac{1}{[G : N]} \sum_{\sigma \in G/N} \sigma \circ \psi_i \circ \sigma^{-1},$$

where σ runs through a system of coset representatives of N in G. Then φ_i is a homomorphism of modules over the subalgebra $A_i'[y_1, \ldots, y_r]$, which commutes with the G-action and has the property $\varphi_i \circ \mu_{g_i} = \mathrm{id}_R$. This completes the proof. □

Remark 1.3. Assume that in the situation of Proposition 1.2 M is a non-zero graded module over $E = \mathrm{End}_{KG}(R)$. Then starting with an M-regular sequence $f_1, \ldots, f_k \in S(W)_+^G$, we can successively append the g_i and use Lemma 1.1 to show that $f_1, \ldots, f_k, g_1, \ldots, g_i$ is M-regular, since φ_i from Proposition 1.2 is a homomorphism of modules over $K[f_1, \ldots, f_k, g_1, \ldots, g_{i-1}]G$, and $(f_1, \ldots, g_m)M \neq M$ by homogeneity. Therefore the above Proposition generalizes Corollary 13 of Campbell et al. [14]. In [14], (almost) the same conclusion on extending regular sequences is reached, but under the hypothesis that G/N not only has order coprime to p, but also is an abelian group. ◁

It is convenient to change the notation slightly at this point by considering $R = S(V^*)$ instead of $S(V)$, since it is better suited for geometric considerations than $S(V)$. Here again G is a finite group, K a field, and V a finite-dimensional KG-module. If $U \leq V$ is a subspace we write

$$U^\perp := \{f \in V^* \mid f|_U = 0\} \leq V^*,$$

and if $A \leq R$ is a K-subalgebra (e.g. $A = R^G$) we write

$$\mathcal{I}_A(U) = (U^\perp)R \cap A. \tag{1.2}$$

If K is algebraically closed, this is the ideal of polynomials from A vanishing on U.

Lemma 1.4. *In the above situation, let $U \leq V$ be a KG-submodule and M a finitely generated graded R^G-module. Then*

$$\operatorname{depth}\left(\mathcal{I}_{S(U^\perp)^G}(U), M\right) = \operatorname{depth}\left(\mathcal{I}_{R^G}(U), M\right).$$

Proof. For simplicity, write $I := \mathcal{I}_{S(U^\perp)^G}(U)$. Since $I \subseteq \mathcal{I}_{R^G}(U)$, the left hand side of the claimed equality is certainly less than or equal to the right hand side. Let $f_1, \ldots, f_d \in I$ be an M-regular sequence of maximal length. We are done if we can show that $f_1, \ldots, f_d$ is maximal as an M-regular sequence in $\mathcal{I}_{R^G}(U)$.

Since $f_1, \ldots, f_d$ is maximal M-regular in I, each $f \in I$ lies in some associated prime ideal $P \in \operatorname{Ass}_{R^G}(M/(f_1, \ldots, f_d)M)$ (see Eisenbud [19, Theorem 3.1b]; notice that we take associated primes in R^G rather than $S(U^\perp)^G$). Hence

$$I \subseteq \bigcup_{P \in \operatorname{Ass}_{R^G}(M/(f_1,\ldots,f_d)M)} (P \cap S(U^\perp)^G).$$

Note that the right hand side is a finite union of prime ideals in $S(U^\perp)^G$. Hence by the prime avoidance theorem (see Eisenbud [19, Lemma 3.3]), I is contained in some $P \in \operatorname{Ass}_{R^G}(M/(f_1, \ldots, f_d)M)$. Since P is an ideal in R^G, also the ideal in R^G generated by I is contained in P:

$$IR^G \subseteq P.$$

If we can show that $\mathcal{I}_{R^G}(U) \subseteq \sqrt{IR^G}$, then $\mathcal{I}_{R^G}(U) \subseteq P$ will follow, so $f_1, \ldots, f_d$ will be maximal M-regular in $\mathcal{I}_{R^G}(U)$, as claimed.

To compare the two ideals, we can assume that K is algebraically closed. For any subset $J \subseteq R$ we write

$$\mathcal{V}_V(J) := \{v \in V \mid f(v) = 0 \ \forall f \in J\}, \tag{1.3}$$

and for a subset $X \subseteq V$ we write

$$\mathcal{I}_{R^G}(X) = \{f \in R^G \mid f|_X = 0\},$$

which is in accordance with the definition of $\mathcal{I}_{R^G}(U)$. If $J \subseteq R^G$ is an ideal, then by Lemma 3.3 in [37] we have

$$\sqrt{J} = \mathcal{I}_{R^G}\left(\mathcal{V}_V(J)\right). \tag{1.4}$$

(This is essentially Hilbert's Nullstellensatz with the slight complication that we are passing from ideals in R^G to varieties in V rather than in the categorical quotient $V/\!\!/G$ and backwards.) Therefore is suffices to show that $\mathcal{V}_V(I) \subseteq \mathcal{V}_V\left(\mathcal{I}_{R^G}(U)\right)$, which follows if we can show that $\mathcal{V}_V(I) \subseteq U$. Take a point $v \in V \setminus U$. Since U is a KG-module, also $\sigma(v) \notin U$ for every $\sigma \in G$, so the set of all $l \in U^\perp$ with $l(\sigma(v)) = 0$ forms a proper subspace of $U^\perp$. Since K is an infinite field, we can select an $l \in U^\perp$ such that $l(\sigma(v)) \neq 0$ for all $\sigma \in G$. Now $f := \prod_{\sigma \in G} \sigma(l)$ lies in $S(U^\perp)^G$ and also in $(U^\perp)R$, so $f \in I$. But

$$f(v) = \prod_{\sigma \in G} l\left(\sigma^{-1}(v)\right) \neq 0.$$

Hence $v \notin \mathcal{V}_V(I)$. This proves that indeed $\mathcal{V}_V(I) \subseteq U$, hence the lemma. $\square$

We can now put things together to obtain the main result of this section. We say that a submodule $U \le V$ of a KG-module V is **non-modular** if the kernel of the G-action on U has index in G which is coprime to the characteristic of K.

Theorem 1.5. *Let G be a finite group, K a field, and V a finite-dimensional KG-module with a non-modular submodule $U \le V$. Set $R := S(V^*)$ and let M be a non-zero graded module over $\operatorname{End}_{KG}(R)$ which is finitely generated over R^G (see the beginning of Section 1.1). Then with $\mathcal{I}_{R^G}(U)$ as defined in (1.2) and regarding M as an R^G-module, we have*

$$\operatorname{depth}(M) = \operatorname{depth}\left(\mathcal{I}_{R^G}(U), M\right) + \dim_K(U).$$

Proof. Set $W := U^\perp \le V^*$. Then a σ in the kernel of the G-action on U acts trivially on the quotient V^*/W, so Proposition 1.2 is applicable here. By Lemma 1.4 the exists an M-regular sequence $f_1, \ldots, f_k \in \mathcal{I}_{S(W)^G}(U)$ of length $k = \operatorname{depth}\left(\mathcal{I}_{R^G}(U), M\right)$. Appending the g_i from Proposition 1.2 and using Lemma 1.1 yields the M-regular sequence $f_1, \ldots, f_k, g_1, \ldots, g_m$, hence

$$\operatorname{depth}(M) \ge k + m = k + \operatorname{codim}_{V^*}(W) = k + \dim_K(U).$$

We claim that $f_1, \ldots, f_k, g_1, \ldots, g_m$ is a maximal M-regular sequence in R^G_+. Indeed, assume that there exists an $f \in R^G_+$ such that $f_1, \ldots, f_k, g_1, \ldots, g_m, f$ is M-regular. By Proposition 1.2(a), the ideal

$$(g_1, \ldots, g_m)R^G + \mathcal{I}_{R^G}(U) \subset R^G$$

has R^G_+ as radical ideal, therefore f^r lies in this ideal for a sufficiently large $r \in \mathbb{N}$. The sequence $f_1, \ldots, f_k, g_1, \ldots, g_m, f^r$ is also M-regular, and remains so if we add an invariant from $(g_1, \ldots, g_m)R^G$ to f^r. Therefore we can assume that f lies in $\mathcal{I}_{R^G}(U)$. Then $f_1, \ldots, f_k, f$ is an M-regular sequence all of whose elements lie in $\mathcal{I}_{R^G}(U)$, in contradiction to $\operatorname{depth}\left(\mathcal{I}_{R^G}(U), M\right) = k$.

Hence $f_1, \ldots, f_k, g_1, \ldots, g_m$ is indeed a maximal M-regular sequence in R^G_+, and the theorem follows. □

Remark 1.6. (a) Choosing $U = V^G$ and $M = R^G$ in Theorem 1.5 and using the fact that the depth can only decrease if one passes to a Sylow p-subgroup yields the lower bound (0.2) of Ellingsrud and Skjelbred [20].

(b) Let $G_p \le G$ be the subgroup of G generated by all elements of G whose order is a power of p. Equivalently, G_p is the smallest normal subgroup containing a Sylow p-subgroup of G, or the smallest normal subgroup such that the order of G/G_p is not divisible by p. Then the largest non-modular submodule $U \le V$ is the fixed space V^{G_p}.

(c) Since the hypothesis of Theorem 1.5 implies that $S(U^*)^G$ is Cohen-Macaulay and since $\mathcal{I}_{R^G}(U)$ is the kernel of the restriction map $R^G \to S(U^*)^G$, one might expect that the equation

$$\operatorname{depth}(R^G) = \operatorname{depth}\left(\mathcal{I}_{R^G}(U), R^G\right) + \operatorname{depth}\left(S(U^*)^G\right) \tag{1.5}$$

holds for any KG-submodule $U \le V$, at least in the case that $S(U^*)^G$ is Cohen-Macaulay. However, this is not true in general, even if U is a direct summand of V, as the following well-known example (see Kemper and Malle [39, Example 2.20]) shows. Consider the action of the additive group $G = G_a$ of $\mathbb{F}_p$ on $W := \mathbb{F}_p^2$ by matrices $\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right)$, and set $V = W \oplus W \oplus W$. By Ellingsrud and Skjelbred [20] (see Equation (0.1)), the invariant ring $S(V^*)^G$ is not Cohen-Macaulay. Let $U \le V$ be the submodule consisting of the first two copies of W. Then $\mathcal{I}_{R^G}(U)$ has height 2 and therefore contains a partial homogeneous system of parameters of length 2. Any partial system of parameters of length at most 2 is R^G-regular (see Landweber and Stong [41]), so

$$\operatorname{depth}\left(\mathcal{I}_{R^G}(U), R^G\right) = 2.$$

Moreover, $S(U^*)^G$ is Cohen-Macaulay, again by Equation (0.1). Therefore

$$\operatorname{depth}\left(\mathcal{I}_{R^G}(U), R^G\right) + \operatorname{depth}\left(S(U^*)^G\right) = 6 > \operatorname{depth}(R^G),$$

so (1.5) becomes false in this example. This shows that Theorem 1.5 indeed depends on the hypothesis that U is a non-modular submodule, and gives a flavor of the subtleness involved in extending regular sequences. ◁

1.2 On the Buchsbaum property

Before we proceed with the discussion of depth, we take a short digression to consider the Buchsbaum property of invariant rings, which is a weakening of the Cohen-Macaulay property. In fact we shall look at the so-called h-Buchsbaum property, which is defined as follows (see Stückrad and Vogel [54, p. 95]): A Noetherian non-negatively graded algebra R is called **h-Buchsbaum** if every homogeneous system of parameters $f_1, \ldots, f_n$ is weakly R-regular, which means that $g f_i \in (f_1, \ldots, f_{i-1})R$ for $1 \leq i \leq n$ and $g \in R$ implies $R_+ g \subseteq (f_1, \ldots, f_{i-1})R$. The Buchsbaum property (which is defined in terms of local rings) implies the h-Buchsbaum property [54, p. 95], and they are equivalent if K is an infinite field by Stückrad and Vogel [54, Chapter 1, Theorem 3.7]. The question of the Buchsbaum property of invariant rings of finite groups was addressed by Nakajima [43] and Campbell et al. [14], who independently proved that if G is a p-group, then $S(V^*)^G$ is Buchsbaum if and only if it is Cohen-Macaulay. In [14], the main tool used for the proof is the result [14, Proposition 20] that if a non-negatively graded algebra R is h-Buchsbaum, then all homogeneous systems of parameters **measure the depth**, i.e., if $f_1, \ldots, f_n$ is a homogeneous system of parameters, then $f_1, \ldots, f_d$ is R-regular for $d = \operatorname{depth}(R)$. Campbell et al. conjectured that for any finite group G and any KG-module V the invariant ring $S(V^*)^G$ is Buchsbaum if and only if it is Cohen-Macaulay. We can use Theorem 1.5 to provide some corroborative evidence for this conjecture by generalizing the result on p-groups mentioned above considerably.

Before stating the generalization, we recall the concept of strongly p-embedded subgroups: A subgroup $H \leq G$ is **strongly p-embedded** for a prime p if the following two conditions hold:

(a) The index $[G : H]$ is coprime to p, and

(b) for $\sigma \in G \setminus H$ the intersection ${}^\sigma H \cap H$ has order coprime to p, where ${}^\sigma H := \sigma H \sigma^{-1}$.

As a typical example, the normalizer of a Sylow p-subgroup P of G is strongly p-embedded if and only if for all $\sigma \in G$ the intersection ${}^\sigma P \cap P$ is either P or the trivial group (i.e., P is a trivial intersection subgroup). It was proved in Kemper [37, Corollary 1.2] that if $H \leq G$ is strongly p-embedded, then

$$\operatorname{depth}(R^G) = \operatorname{depth}(R^H),$$

where R is a non-negatively graded Noetherian algebra over a field of characteristic p on which G acts. In fact, it is shown in the proof of [37, Corollary 1.2] that a sequence $f_1, \ldots, f_r \in R^G$ is R^G-regular if and only if it is R^H-regular.

Theorem 1.7. *Let G be a finite group, K a field of characteristic p, and V a finite-dimensional KG-module. Assume that there exists a strongly p-embedded subgroup $H \leq G$ such that $V^{H_p} \neq 0$, where H_p is defined in Remark 1.6(a). Then with $R = S(V^*)$ we have the equivalence*

$$R^G \text{ is Buchsbaum} \iff R^G \text{ is Cohen-Macaulay.}$$

Proof. Set $U := V^{H_p}$. By Kemper [37, Corollary 1.2] and Theorem 1.5 we have

$$\operatorname{depth}(R^G) = \operatorname{depth}(R^H) = \operatorname{depth}\left(\mathcal{I}_{R^H}(U), R^H\right) + \dim_K(U). \tag{1.6}$$

We distinguish two cases. First, we consider the case where $\operatorname{depth}\left(\mathcal{I}_{R^H}(U), R^H\right) = \operatorname{codim}_V(U)$. Then by (1.6) we have $\operatorname{depth}(R^G) = \dim_K(V) = \dim(R^G)$, so R^G is Cohen-Macaulay. On the other hand, assume that $\operatorname{depth}\left(\mathcal{I}_{R^H}(U), R^H\right) < \operatorname{codim}_V(U)$. Since $\mathcal{I}_{R^G}(U) \subseteq \mathcal{I}_{R^H}(U)$, we also have

$$\operatorname{depth}\left(\mathcal{I}_{R^G}(U), R^H\right) < \operatorname{codim}_V(U).$$

Since $\mathcal{I}_{R^G}(U) = (U^{\perp})R \cap R^G$ and R is integral over R^G, it follows from Eisenbud [19, Proposition 9.2] that

$$\operatorname{ht}\left(\mathcal{I}_{R^G}(U)\right) = \operatorname{ht}((U^{\perp})R) = \operatorname{codim}_V(U).$$

Therefore there exists a partial homogeneous system of parameters $f_1, \ldots, f_r \in \mathcal{I}_{R^G}(U)$ with $r = \operatorname{depth}\left(\mathcal{I}_{R^G}(U), R^H\right) + 1$ (see Eisenbud [19, Theorem 13.3]). This sequence cannot be R^H-regular, hence by the strong p-embeddedness it is not R^G-regular, either. But using (1.6) and the hypothesis $\dim_K(U) > 0$ we obtain

$$r \leq \operatorname{depth}\left(\mathcal{I}_{R^H}(U), R^H\right) + 1 \leq \operatorname{depth}(R^G).$$

Hence the sequence can be completed to a homogeneous system of parameters which does not measure the depth (of R^G), and it follows from Campbell et al. [14, Proposition 20] that R^G is not h-Buchsbaum and therefore not Buchsbaum.

We have seen that R^G is either Cohen-Macaulay or not Buchsbaum, hence the claimed equivalence. □

Example 1.8. To make the application of Theorem 1.7 easier, we list a few simple conditions under which the hypothesis is satisfied:

- G has a normal Sylow p-subgroup (since then $G_p = P$ is the Sylow subgroup, and hence $V^{G_p} \neq 0$).
- G has order not divisible by p^2 (since then the normalizer H of a Sylow p-subgroup is strongly p-embedded, and $V^{H_p} \neq 0$ by the above).
- $V^G \neq 0$. For example, this is always the case if G acts by permutations of a basis of V. ◁

2 Cohomological methods

We now turn back to the original topic of the depth of an invariant ring. As it stands, Theorem 1.5 only shifts the question of $\operatorname{depth}(M)$ to the question of $\operatorname{depth}\left(\mathcal{I}_{R^G}(U), M\right)$ (where M could stand for the invariant ring R^G), which seems equally intractable. However, we shall see in this section that by using methods from cohomology one can actually say something about the latter depth in many cases.

2.1 The support of cohomology

We fix some notation. Let G be a finite group and K a field of positive characteristic p. Furthermore, let R be a Noetherian K-algebra (associative, commutative and with unity) with an action of G by K-automorphisms. We can look at a somewhat more general situation by taking a non-zero, finite-dimensional KG-module W and considering the tensor product $M := W \otimes_K R$. If $R = S(V^*)$ with a KG-module V, then M can be interpreted as the set of polynomial functions $V \to W$, and the invariants in M^G are the G-equivariant functions. If W is one-dimensional, then M^G is the set of relative G-invariants in R with respect to the linear character given by W^*. As we have seen at the beginning of Section 1.1, M^G and, more generally, the cohomology $H^*(G, M)$ are modules over $E := \operatorname{End}_{KG}(R)$. Therefore the results of Section 1.1 are applicable to M^G.

Being an E-module, $H^*(G,M)$ also is a module over R^G. More explicitly, an invariant $f \in R^G$ acts on $H^*(G,M)$ by the map induced by $M \to M$, $w \otimes g \mapsto w \otimes fg$. Fix an $\alpha \in H^*(G,R)$. Then we write

$$I_\alpha := \operatorname{Ann}_{R^G}(\alpha) = \{f \in R^G \mid f\alpha = 0\}.$$

This is an ideal in R^G. We will be particularly interested in the E-submodule

$$H^+(G,M) := \bigoplus_{i>0} H^i(G,M) \tag{2.1}$$

and its annihilator as an R^G-module. The relevance of cohomology and in particular of the ideal I_α in the study of depth is given by Theorem 1.9 of Kemper [36] (or [37, Theorem 1.4] for $M = R$), which says the following: If $H^r(G,M) \neq 0$ for $r > 0$ and if r is the smallest positive number with this property, then

(a) If $f_1, \ldots, f_k \in R^G$ are invariants with $(f_1, \ldots, f_k)M \neq M$ forming an R-regular sequence and $k \leq r+1$, then $f_1, \ldots, f_k$ is M^G-regular.

(b) Any R-regular sequence $f_1, \ldots, f_{r+2} \in I_\alpha$ of $r+2$ (or more) invariants in I_α is not M^G-regular.

It follows that if R is Cohen-Macaulay (e.g., $R = S(V^*)$) and $I_\alpha M \neq M$, then

$$\operatorname{depth}(I_\alpha, M^G) = \min\{r+1, \operatorname{ht}(I_\alpha)\}.$$

It is this formula which makes Theorem 1.5 applicable.

We specialize our situation by considering a finite-dimensional KG-module V and taking $R := S(V^*)$ to be the symmetric algebra of the dual. We assume that K is algebraically closed. (This assumption is made here for reasons of convenience, and will become a without-loss assumption; alternatively, we could use the language of schemes.) For $\alpha \in H^*(G,M)$, we consider the variety

$$\mathcal{V}_\alpha := \mathcal{V}_V(I_\alpha),$$

with $\mathcal{V}_V(J)$ as defined in (1.3). Let us call $\mathcal{V}_\alpha$ the **support** of α. Notice that although I_α is an ideal in R^G and not in R, we nevertheless take the associated variety in V rather than in the categorical quotient $V/\!\!/G$. This avoids unnecessary complications. With the subgroup G_p and its fixed space V^{G_p} as introduced in Remark 1.6(a), Theorem 1.5 now leads to the following proposition.

Proposition 2.1. *With the above notation, assume that there exists a non-zero $\alpha \in H^r(G,M)$ such that $\mathcal{V}_\alpha \subseteq V^{G_p}$, where r is the smallest positive number such that $H^r(G,M) \neq 0$. Then*

$$\operatorname{depth}(M^G) = \min\left\{\dim_K(V^{G_p}) + r + 1, \dim(M^G)\right\}.$$

Proof. If $M^G = 0$ then the claimed equation reads $-1 = -1$. Hence we can assume that $M^G \neq 0$, which implies $\dim(M^G) = \dim_K(V)$ since $\operatorname{Ann}_{R^G}(M^G) = 0$. Set $U := V^{G_p}$. From the hypothesis it follows with (1.4) that

$$\mathcal{I}_{R^G}(U) \subseteq \sqrt{I_\alpha}.$$

Since $\operatorname{depth}(\sqrt{I_\alpha}, M^G) = \operatorname{depth}(I_\alpha, M^G)$, it follows from the statements (a) and (b) above that

$$\operatorname{depth}\left(\mathcal{I}_{R^G}(U), M^G\right) = \min\left\{r+1, \operatorname{ht}\left(\mathcal{I}_{R^G}(U)\right)\right\} = \min\left\{r+1, \operatorname{codim}_V(U)\right\}.$$

Now the proposition follows from Theorem 1.5. □

Proposition 2.1 prompts us to search for non-zero elements $\alpha \in H^*(G,M)$ whose support $\mathcal{V}_\alpha$ is contained in the fixed space V^{G_p}. The main tool for the investigation of annihilators (and thus of supports) of elements in the cohomology will be the **transfer** map

$$\operatorname{Tr}_G \colon R \to R^G,\ f \mapsto \sum_{\sigma \in G} \sigma(f),$$

and its relative versions (see below). What makes the transfer useful in for our purposes is that we know its locus (see Equation (2.3)). We will prove that an invariant in the image of the transfer annihilates every $\alpha \in H^+(G, M)$. For this purpose, we first prove two preliminary results. The first of these, which is purely representation theoretic, is more or less well-known and was proved, in greater generality and apparently for the first time, by Osima [46, Theorem 4]. For the convenience of the reader, I give a short proof here which was shown to me by Ian Hughes.

Proposition 2.2. *Let V be an indecomposable module over KG, where K is a field and G is a finite group. Write*

$$s := \sum_{\sigma \in G} \sigma \in KG.$$

Then $sV \neq 0$ implies that V is a projective KG-module.

Proof. There exists a projective module P with an epimorphism $\varphi\colon P \to V$ of KG-modules. Let

$$P = U_1 \oplus \cdots \oplus U_r$$

be the decomposition of P into indecomposable modules. By assumption, there exists $v \in V$ such that $sv \neq 0$. Let $u \in P$ be a preimage of v under φ, and $u = u_1 + \cdots + u_r$ with $u_i \in U_i$. Then $\varphi(su) = sv \neq 0$. Therefore $\varphi(su_i) \neq 0$ for some i, which we can assume to be equal to 1. Hence $K \cdot su_1$ is a submodule of U_1 which is isomorphic to the trivial module K. Since U_1 is an indecomposable projective module, its socle is simple (see Alperin [2, Corollary 6.5]), so $K \cdot su_1$ is the socle. We claim that the restriction

$$\varphi|_{U_1}\colon U_1 \to V$$

is injective. Indeed, if the kernel were non-zero, it would contain the socle of U_1, but $\varphi(su_1) \neq 0$. Since as a projective module U_1 is also injective (see Alperin [2, Theorem 6.4]), the above mapping splits, so U_1 is a direct summand of V. But V was assumed indecomposable, hence $V \cong U_1$, which is projective. □

In order to formulate the following result, we need some additional notation. If U and V are two KG-modules and $u \in U^G$ is an invariant, then the KG-homomorphism $V \to U \otimes_K V$, $v \mapsto u \otimes v$ induces a homomorphism $H^i(G, V) \to H^i(G, U \otimes_K V)$. For $\alpha \in H^i(G, V)$, we denote the image under this homomorphism by $u \otimes \alpha$. Furthermore, if $N \trianglelefteq G$ is a normal subgroup, we denote the inflation map $H^i(G/N, V^N) \to H^i(G, V)$ by $\inf_{G/N}$ (see Benson [4, p. 64]).

Proposition 2.3. *Let G be a finite group and $N \trianglelefteq G$ a normal subgroup. Furthermore, let K be a field and U, V two (not necessarily finite-dimensional) KG-modules. We write $s_{G/N} = \sum_{\sigma \in G/N} \sigma$, which is an element of the group ring $K(G/N)$. Then for $\alpha \in H^i(G/N, V^N)$ with $r > 0$ and for $u \in s_{G/N} U^N$ we have*

$$u \otimes \inf_{G/N}(\alpha) = 0$$

as an element of $H^i(G, U \otimes_K V)$.

Proof. There is a finite-dimensional $K(G/N)$-submodule U' of U^N such that $u \in s_{G/N} U'$. Take a decomposition $U' = U_1 \oplus \cdots \oplus U_r$ with U_j indecomposable $K(G/N)$-modules and write $u = u_1 + \cdots + u_r$ with $u_j \in U_j$. Then those U_j with $u_j \neq 0$ are projective by Proposition 2.2. Therefore u lies in a finite-dimensional, projective $K(G/N)$-submodule P of U^N.

Since α is given by a homomorphism from a module in a projective resolution of K (as a $K(G/N)$-module) into V^N, there is also a finite-dimensional $K(G/N)$-submodule $V' \leq V^N$ with $\alpha \in H^i(G/N, V')$. (It would be more exact to say that α lies in the image of the map $H^i(G/N, V') \to H^i(G/N, V^N)$ induced by the inclusion.) Since the tensor product of a projective module with any module is again projective (see Alperin [2, Lemma 7.4]), $P \otimes_K V'$ is also projective. Hence $H^i(G/N, P \otimes_K V') = 0$ for $r > 0$, and in particular $u \otimes \alpha = 0$ as an element in $H^i(G/N, P \otimes_K V')$.

If we consider $u \otimes \alpha$ as an element in $H^i(G/N, (U \otimes_K V)^N)$ (i.e., we enlarge the module), it will still be zero. Now we apply the inflation map and obtain

$$u \otimes \inf_{G/N}(\alpha) = \inf_{G/N}(u \otimes \alpha) = 0,$$

where the inflation on the right hand side is $\inf_{G/N}: H^i(G/N, (U \otimes_K V)^N) \to H^i(G, U \otimes_K V)$. $\square$

Now we return to the situation where a finite group G acts on $M = W \otimes_K R$ as above, with R a K-algebra and W a finite-dimensional KG-module. If $N \trianglelefteq G$ is a normal subgroup, we write $I_{\mathrm{Tr}_{G/N}} \subseteq R^G$ for the image of the **relative transfer**

$$\mathrm{Tr}_{G/N}: R^N \to R^G, \ f \mapsto \sum_{\sigma \in G/N} \sigma(f).$$

Note that $I_{\mathrm{Tr}_{G/N}}$ is an ideal in R^G.

Corollary 2.4. *In the above situation, let $\alpha \in H^i(G, M)$ be in the image of the inflation map*

$$\inf_{G/N}: H^i(G/N, M^N) \to H^i(G, M),$$

with $r > 0$. Then

$$I_{\mathrm{Tr}_{G/N}} \subseteq I_\alpha,$$

where, as above, I_α is the ideal in R^G of all invariants f with $f\alpha = 0$.

Proof. Take $f \in I_{\mathrm{Tr}_{G/N}}$. Then by Proposition 2.3 we have

$$f \otimes \alpha = 0$$

as an element in $H^i(G, R \otimes_K M)$. Now we apply the map induced by $R \otimes_K M \to R$, $g \otimes m \mapsto gm$, which is clearly a homomorphism of KG-modules. This sends the right hand side to zero, and the left hand side to $f\alpha \in H^i(G, R)$, since tensoring by f and then applying the above map is simply multiplication with f. Therefore we obtain $f \in I_\alpha$. $\square$

Again we specialize to the situation where $R = S(V^*)$ with V a KG-module. Assuming that K is algebraically closed, we define the **relative transfer locus** to be the variety

$$\mathcal{V}_{\mathrm{Tr}_{G/N}} = \mathcal{V}_V(I_{\mathrm{Tr}_{G/N}})$$

in V. Corollary 2.4 implies that

$$\mathcal{V}_\alpha \subseteq \mathcal{V}_{\mathrm{Tr}_{G/N}}. \tag{2.2}$$

for any $\alpha \in H^+(G, M)$ lying in the image of the inflation map. On the other hand, the relative transfer locus is well known by the papers of Fleischmann [24] or Campbell et al. [14, Theorem 7]. The result is

$$\mathcal{V}_{\mathrm{Tr}_{G/N}} = \bigcup_{\substack{\sigma \in G, \\ \mathrm{ord}(\sigma N) = p}} V^\sigma. \tag{2.3}$$

Here the union runs over all $\sigma \in G$ such that the order of the coset σN (as an element of G/N) is p. Hence in order to apply Proposition 2.1, all we have to do is choose a normal subgroup $N \trianglelefteq G$ such that the above union of subspaces will be contained in V^{G_p}.

Definition 2.5. *Let G be a finite group, K a field of characteristic $p > 0$, and V a KG-module. Then we write $G_{V\text{-sing}}$ for the subgroup of G generated by all $\sigma \in G$ of order a power of p such that*

$$V^\sigma \supsetneqq V^{G_p}.$$

Here G_p denotes the subgroup of G generated by all elements of order a power of p (as in Remark 1.6(b)).

It is clear that $G_{V\text{-sing}}$ is a normal subgroup of G. We can now prove the main theorem of this section.

Theorem 2.6. *Let G be a finite group, K a field of positive characteristic p, and V a finite-dimensional KG-module. Take a Sylow p-subgroup P of G and $G_{V\text{-sing}}$ as in Definition 2.5. Furthermore, let $R = S(V^*)$ be the symmetric algebra of the dual and $M := W \otimes_K R$ with W a finite-dimensional KG-module (see the beginning of Section 2.1). Assume that there exists a positive integer r such that $H^i(G, M) = 0$ for $0 < i < r$ and the inflation*

$$\inf_{G/G_{V\text{-sing}}} \colon H^r(G/G_{V\text{-sing}}, M^{G_{V\text{-sing}}}) \to H^r(G, M)$$

is not the zero-map. Then

$$\operatorname{depth}(M^G) = \min\left\{\dim_K(V^P) + r + 1, \dim(M^G)\right\}.$$

Remark. If $M^G \neq 0$, then we have $\dim(M^G) = \dim_K(V)$. Indeed, $M^G \neq 0$ implies $\operatorname{Ann}_{R^G}(M^G) = 0$ and therefore $\dim(M^G) = \dim(R^G) = \dim_K(V)$ by Equation (0.6). On the other hand, if $M^G = 0$, then $\dim(M^G) = -1$. ◁

Proof of Theorem 2.6. First we remark that the depth of M^G does not change if the ground field K is extended. This follows from the characterization of depth in terms of the spaces $\operatorname{Ext}^i_{R^G}(K, M^G)$ (see Eisenbud [19, Proposition 18.4]). The hypotheses and the right hand side of the claimed formula are also unchanged under extensions of K. Hence we can assume that K is algebraically closed.

By assumption there exists a non-zero $\alpha \in H^r(G, M)$ in the image of $\inf_{G/G_{V\text{-sing}}}$. By (2.2) and (2.3) we have

$$\mathcal{V}_\alpha \subseteq \bigcup_{\substack{\sigma \in G,\\ \operatorname{ord}(\sigma G_{V\text{-sing}}) = p}} V^\sigma.$$

We want to show that the union is contained in V^{G_p}, so take a $\sigma \in G$ with $\operatorname{ord}(\sigma G_{V\text{-sing}}) = p$. Write the order of σ (as an element in G) as mq with q a power of p and $p \nmid m$. Then $V^\sigma \subseteq V^{\sigma^m}$, but σ^m is an element of order a power of p which is not contained in $G_{V\text{-sing}}$. Therefore $V^{\sigma^m} = V^{G_p}$. Hence we obtain

$$V^\sigma \subseteq V^{\sigma^m} = V^{G_p}.$$

This shows that the above union is indeed contained in V^{G_p}, thus $\mathcal{V}_\alpha \subseteq V^{G_p}$. Now Proposition 2.1 yields

$$\operatorname{depth}(M^G) = \min\left\{\dim_K(V^{G_p}) + r + 1, \dim(M^G)\right\}.$$

With the following Lemma 2.7, this completes the proof. □

Lemma 2.7. *With the notation of Theorem 2.6 assume that*

$$H^+(G/G_{V\text{-sing}}, M^{G_{V\text{-sing}}}) \neq 0,$$

where H^+ is defined by (2.1). Then

$$V^{G_p} = V^P.$$

Proof. Since P is contained in G_p, the inclusion $V^{G_p} \subseteq V^P$ is clear. By way of contradiction, assume that the inclusion is strict. Then for every $\sigma \in P$ we have $V^{G_p} \subsetneqq V^\sigma$, hence $\sigma \in G_{V\text{-sing}}$. Therefore the quotient group $G/G_{V\text{-sing}}$ is of order coprime to p, which implies $H^+(G/G_{V\text{-sing}}, M^{G_{V\text{-sing}}}) = 0$. This contradicts the hypothesis. □

Remark 2.8. We make a few remarks which mainly apply to the situation where $M = R$.

(a) In the situation of the Theorem (with $M = R$), K appears as the degree-0 component of R, which is a direct summand. Hence we have an embedding $H^i(G, K) \hookrightarrow H^i(G, R)$. If the group order $|G|$ is divisible by p, then we know that $H^i(G, K) \neq 0$ for some positive i (see Benson [5, Theorem 4.1.3]). Therefore there exists a unique minimal positive r such that $H^r(G, R) \neq 0$, and it is for this r that the hypothesis of Theorem 2.6 has to be tested.

(b) If $r = 1$, then we have an exact sequence

$$0 \to H^1(G/G_{V\text{-sing}}, M^{G_{V\text{-sing}}}) \to H^1(G, M) \to H^1(G_{V\text{-sing}}, M)^{G/G_{V\text{-sing}}} \to \cdots,$$

where the second map is $\inf_{G/G_{V\text{-sing}}}$ (see Evens [21, Corollary 7.2.3]. Therefore the hypothesis of Theorem 2.6 reads

$$H^1(G/G_{V\text{-sing}}, M^{G_{V\text{-sing}}}) \neq 0$$

in this case.

For example, if G is a p-group and $M = R$, then $r = 1$, and $H^1(G/G_{V\text{-sing}}, R^{G_{V\text{-sing}}}) \neq 0$ if and only if $G_{V\text{-sing}} \neq G$. Moreover, $G_p = G$. Hence in the case of a p-group G and $M = R$, the hypothesis is met if and only if G is *not* generated by elements $\sigma \in G$ with $V^\sigma \supsetneqq V^G$.

(c) In Campbell et al. [14], the authors showed that if a linear group $G \leq \mathrm{GL}(V)$ is shallow, then

$$\mathrm{depth}(R^G) = \min\left\{\dim_K(V^P) + 2, \dim_K(V)\right\}.$$

To recall the definition of a shallow linear group given in [14], let G'_p be the subgroup generated by a Sylow p-subgroup P and the commutator subgroup G'. G is said to be **shallow** if it possesses a normal subgroup N of index p such that all $\sigma \in G'_P$ with

$$V^\sigma \supsetneqq V^{G'_p} \tag{2.4}$$

lie in N. The above equation says that if G acts as a shallow group on V, then (G, V) is flat (as defined on page 3). Important examples of shallow groups are abelian groups with cyclic Sylow p-subgroups. The condition of shallowness can be expressed as

$$H^1(G/G'_{V\text{-sing}}, K) \neq 0,$$

where $G'_{V\text{-sing}}$ is the subgroup generated by all $\sigma \in G'_P$ satisfying (2.4). Therefore shallowness implies the hypothesis of Theorem 2.6, so Theorem 2.6 is a generalization of the result of Campbell et al. [14], which in turn generalizes the main result of Ellingsrud and Skjelbred [20]. More precisely, we have a generalization in four ways:

(1) Higher cohomology is taken into account.

(2) It is not necessary to have non-zero cohomology with values in the ground field.

(3) The subgroup G_P is smaller than G'_P, hence $G_{V\text{-sing}}$ is smaller than $G'_{V\text{-sing}}$. Therefore it is easier for the inflation map $\inf_{G/G_{V\text{-sing}}}$ to be non-zero than for $\inf_{G/G'_{V\text{-sing}}}$.

(4) The case of equivariants (i.e., $M \neq R$) is also covered.

However, if G is a p-group, the hypothesis of Theorem 2.6 is met if and only if G is shallow. ◁

The scope of Theorem 2.6 can be considerably widened by passing to a strongly p-embedded subgroup H (see the discussion on page 12). If $H \leq G$ is strongly p-embedded, then by Kemper [36, Proposition 1.21 and Theorem 1.23] (or [37, Corollary 1.2] for the case $M = R$)

$$\mathrm{depth}(M^G) = \mathrm{depth}(M^H).$$

In fact, this equality was derived from the fact that for any KG-module U and for $i > 0$, the restriction map $H^i(G,U) \to H^i(H,U)$ is an isomorphism. In particular, in the situation of Theorem 2.6 the minimal positive r with $H^r(G,M) \neq 0$ will also be the minimal positive r with $H^r(H,M) \neq 0$. We obtain the following generalization of Theorem 2.6.

Corollary 2.9. *With the notation of Theorem 2.6 assume that there exists a strongly p-embedded subgroup $H \leq G$ such that the inflation map*

$$\inf_{H/H_{V\text{-sing}}}: H^r(H/H_{V\text{-sing}}, M^{H_{V\text{-sing}}}) \to H^r(H,M)$$

is non-zero (where $H^i(H,M) = 0$ for $0 < i < r$). Then

$$\operatorname{depth}(M^G) = \min\left\{\dim_K(V^P) + r + 1, \dim(M^G)\right\}.$$

Proof. The statement is correct for $M^G = 0$. So we can assume $M^G \neq 0$, which implies $M^H \neq 0$ and $\dim(M^G) = \dim(M^H)$. By Kemper [36, Proposition 1.21 and Satz 1.23], $\operatorname{depth}(M^G) = \operatorname{depth}(M^H)$. Since H is strongly p-embedded, it contains a Sylow p-subgroup of G, and since $\dim_K(V^P)$ is the same for all Sylow p-subgroups, we can assume that $P \leq H$. Now Theorem 2.6 says that

$$\operatorname{depth}(M^H) = \min\left\{\dim_K(V^P) + r + 1, \dim(M^H)\right\}.$$

□

Although only an upper bound for the support of $H^+(G,R)$ (see (2.1)) is required for the purposes of this paper, it is if some interest to determine this support exactly. We will do this in the more general setting of group actions on algebras. More precisely, suppose that a finite group G acts on a finitely generated algebra R over a field K of characteristic p by automorphisms fixing K. For $x \in X := \operatorname{Spec}(R)$ we define the "inertia group" to be

$$G_x := \{\sigma \in G \mid (\sigma - 1)R \subseteq x\}.$$

Moreover, let

$$\pi_G \colon X \to X/\!\!/G := \operatorname{Spec}(R^G)$$

be the morphism given by intersecting a prime ideal in R with R^G.

Theorem 2.10. *With the above notation we have*

$$\operatorname{Supp}\left(H^+(G,R)\right) = \pi_G\left(\{x \in X \mid p \text{ divides } |G_x|\}\right).$$

Proof. We have already seen in Corollary 2.4 that $I_{\mathrm{Tr}_G} \subseteq \operatorname{Ann}_{R^G}(H^+(G,R))$. Therefore

$$\operatorname{Supp}\left(H^+(G,R)\right) \subseteq \mathcal{V}_{\mathrm{Tr}_G}.$$

By Kemper [36, Proposition 4.5 and Satz 4.7] (see also Feshbach [22]) we have

$$\mathcal{V}_{\mathrm{Tr}_G} = \pi_G\left(\{x \in X \mid p \text{ divides } |G_x|\}\right) =: Y.$$

This proves that $\operatorname{Supp}\left(H^+(G,R)\right) \subseteq Y$.

For the reverse inclusion, take any $\pi_G(x) \in Y$. By the definition of Y, there exists a cyclic subgroup $H = \langle\tau\rangle \subseteq G_x$ of order p. Using the Evens norm from $H^*(H,K)$ into $H^*(G,K)$, we find an element $\alpha \in H^i(G,K)$ with i positive and even such that $\operatorname{res}_{G,H}(\alpha) \neq 0$ (see Benson [5, Theorem 4.1 and its proof]). Let $\bar\alpha$ be the image of α under the map $H^i(G,K) \to H^i(G,R)$ induced by the embedding $K \subseteq R$. Take any $f \in \operatorname{Ann}_{R^G}(H^+(G,R))$. Then $f\bar\alpha = 0$ and therefore

$$f \operatorname{res}_{G,H}(\bar\alpha) = \operatorname{res}_{G,H}(f\bar\alpha) = 0.$$

But we have

$$H^i(H,R) \cong R^H/(\tau-1)^{p-1}R$$

(see Evens [21, p. 6] or (3.1)). Since $\operatorname{res}_{G,H}(\bar{\alpha})$ is represented by a non-zero element from K, it follows that $f \in (\tau-1)^{p-1}R \subseteq (\tau-1)R$. Since f was taken as an arbitrary element from $\operatorname{Ann}_{R^G}(H^+(G,R))$, we conclude that

$$\operatorname{Ann}_{R^G}(H^+(G,R)) \subseteq (\tau-1)R \subseteq x.$$

Thus we have $\pi(x) \in \operatorname{Supp}(H^+(G,R))$ as desired. □

Remark 2.11. Theorem 2.10 also provides a geometric criterion to decide whether $H^+(G,R)=0$. For example, if a cyclic group $G=\langle\sigma\rangle \cong Z_p$ acts on a finite-dimensional vector space V over a field of characteristic p by $\sigma(v)=v+v_0$ with $v_0 \neq 0$ a fixed vector, then $H^+(G,S(V^*))=0$. Of course, the action is not linear. It follows by Kemper [37, Theorem 1.4] that $S(V^*)^G$ is Cohen-Macaulay. ◁

2.2 The depth of cohomology modules

Our main interest lies in the depth of M^G, which is $H^0(G,M)$. Of course one might as well ask about the depth of $H^i(G,M)$ (as an R^G-module) for $i>0$. This question was considered for the cyclic group of order p by Ellingsrud and Skjelbred [20].

Proposition 2.12. *Let G be a finite group acting by K-automorphisms on a Noetherian non-negatively graded algebra R over a field K of characteristic p, and let W be a finite-dimensional KG-module. Then with $M := W \otimes_K R$ we have for all non-negative i:*

(a) If $H \leq G$ is a subgroup such that $p \nmid [G:H]$ and if $H^i(G,M) \neq 0$, then

$$\operatorname{depth}\left(H^i(G,M)\right) \geq \operatorname{depth}\left(H^i(H,M)\right).$$

(b) If $H \leq G$ is a strongly p-embedded subgroup, then

$$\operatorname{depth}\left(H^i(G,M)\right) = \operatorname{depth}\left(H^i(H,M)\right).$$

Here $H^i(G,M)$ and $H^i(H,M)$ are considered as modules over R^G and R^H, respectively.

Proof. If H is strongly p-embedded, then the restriction map $H^i(G,M) \to H^i(H,M)$ is an isomorphism of R^G-modules, so $\operatorname{depth}\left(R^G_+, H^i(G,M)\right) = \operatorname{depth}\left(R^G_+, H^i(H,M)\right)$. Since R^H is integral over R^G, we also have $\operatorname{depth}\left(R^G_+, H^i(H,M)\right) = \operatorname{depth}\left(R^H_+, H^i(H,M)\right)$ (see Kemper [36, Proposition 1.17]). From this the statement (b) follows.

To prove (a), let $f_1,\ldots,f_k \in R^G_+$ be an $H^i(H,M)$-regular sequence with $k = \operatorname{depth}(R^G_+, H^i(H,M)) = \operatorname{depth}\left(R^H_+, H^i(H,M)\right)$. We want to see that $f_1,\ldots,f_k$ is also $H^i(G,M)$-regular. So assume

$$f_1\alpha_1 + \cdots + f_l\alpha_l = 0 \quad \text{with} \quad \alpha_j \in H^i(G,M)$$

for some $l \leq k$. Applying the restriction map $\operatorname{res}\colon H^i(G,M) \to H^i(H,M)$ yields

$$f_1 \operatorname{res}(\alpha_1) + \cdots + f_l \operatorname{res}(\alpha_l) = 0.$$

By the $H^i(H,M)$-regularity this implies

$$\operatorname{res}(\alpha_l) = f_1\beta_1 + \cdots + f_{l-1}\beta_{l-1} \quad \text{with} \quad \beta_j \in H^i(H,M).$$

It is easily seen from the definition of corestriction (see Benson [4, p. 67], where the names trace or transfer are used for the corestriction) that the map

$$\varphi := \frac{1}{[G:H]} \operatorname{cores}\colon H^i(H,M) \to H^i(G,M)$$

commutes with multiplication by an f_j, and $\varphi \circ \mathrm{res} = \mathrm{id}$ (Benson [4, Proposition 3.6.17]). Applying φ to the above equation, we obtain

$$\alpha_l = f_1\varphi(\beta_1) + \cdots + f_{l-1}\varphi(\beta_{l-1}) \in (f_1, \ldots, f_{l-1})H^i(G, M).$$

This shows that $f_1, \ldots, f_k$ is $H^i(G, M)$-regular. □

The following result (in particular the statement (c)) may be somewhat surprising.

Theorem 2.13. *In the setting of Theorem 2.6 (but without any assumptions about cohomology spaces), let i be a positive integer. Then we have:*

(a) If $H^i(G, M) \neq 0$, then for a Sylow p-subgroup $P \leq G$ we have

$$\operatorname{depth}\left(H^i(G, M)\right) \geq \dim_K(V^P).$$

(b) If there exists a strongly p-embedded subgroup $H \leq G$ such that the inflation map

$$\inf_{H/H_{V\text{-sing}}} : H^i(H/H_{V\text{-sing}}, M^{H_{V\text{-sing}}}) \to H^i(H, M)$$

is non-zero, then

$$\operatorname{depth}\left(H^i(G, M)\right) = \dim_K(V^P).$$

(c) If there exists a strongly p-embedded subgroup $H \leq G$ such that $H_{V\text{-sing}}$ is the trivial group, then $H^i(G, M)$ is Cohen-Macaulay as an R^G-module.

Proof. We first prove statement (a). Indeed, $H^i(P, M)$ is non-zero since the restriction provides an embedding $H^i(G, M) \hookrightarrow H^i(P, M)$. Furthermore, $H^i(P, M)$ is a module over $\mathrm{End}_{KP}(R)$ (see the beginning of Section 1.1) and $V^P \leq V$ is a non-modular KP-submodule. Hence Theorem 1.5 yields

$$\operatorname{depth}\left(H^i(P, M)\right) \geq \dim_K(V^P),$$

from which (a) follows by using Proposition 2.12(a).

Next we prove (b) and (c). As in the proof of Theorem 2.6 we can assume that K is algebraically closed. Assume the hypothesis of (b) and take $\alpha \neq 0$ in the image of the inflation. Then multiplication with an element from $I_\alpha = \mathrm{Ann}_{R^H}(\alpha)$ is not injective on $H^i(H, M)$, hence

$$\operatorname{depth}\left(I_\alpha, H^i(H, M)\right) = 0.$$

But from (2.2) and (2.3) and by Lemma 2.7 we see that

$$\mathcal{V}_\alpha \subseteq V^{H_p} = V^P.$$

By (1.4) it follows that $I_{R^H}(V^P) \subseteq \sqrt{I_\alpha}$, so

$$\operatorname{depth}\left(I_{R^H}(V^P), H^i(H, M)\right) = 0.$$

The statement (b) now follows from Theorem 1.5 and Proposition 2.12(b).

In the situation of (c) we can assume that $H^i(H, M) \neq 0$, since otherwise the depth and the dimension are both equal to -1. So the hypothesis of statement (b) is satisfied, and we obtain $\operatorname{depth}\left(H^i(H, M)\right) = \dim_K(V^P)$. On the other hand,

$$I_{\mathrm{Tr}_H} \subseteq \mathrm{Ann}_{R^H}\left(H^i(H, M)\right)$$

by Corollary 2.4. Hence for the Krull dimension of $H^i(H, M)$ we have

$$\dim\left(H^i(H, M)\right) \leq \dim\left(R^H / I_{\mathrm{Tr}_H}\right).$$

But $\mathcal{V}_V(I_{\mathrm{Tr}_H}) \subseteq V^P$ (see the proof of Theorem 2.6), hence

$$\dim\left(H^i(H, M)\right) \leq \dim_K(V^P) = \operatorname{depth}\left(H^i(H, M)\right).$$

Therefore the depth is equal to the dimension, and (c) follows. □

Example 2.14. If the order of G is not divisible by p^2, then the normalizer $H := \mathcal{N}_G(P)$ of a Sylow p-subgroup of G is strongly p-embedded, and $H_{V\text{-sing}}$ is trivial. Therefore by Theorem 2.13(c) every $H^i(G, M)$ with $i > 0$ is Cohen-Macaulay. This is in sharp contrast to the situation for $H^0(G, M) = M^G$. For example, if $G = Z_p$, the cyclic of order p, then by Theorem 2.6, M^G is Cohen-Macaulay if and only if $\dim_K(V^P) \geq \dim_K(V) - 2$ (see also Ellingsrud and Skjelbred [20] for the case $M = R$). The fact that $H^i(G, R)$ is Cohen-Macaulay for G cyclic of order p was also proved by Ellingsrud and Skjelbred [20]. ◁

2.3 Transporting cohomology from subgroups

For applications of Corollary 2.9 the most common situation is that the strongly p-embedded subgroup H is the normalizer of a Sylow p-subgroup P of G. Then $H_p = P$ and $H_{V\text{-sing}} \leq P$, so $H/H_{V\text{-sing}}$ has the normal Sylow p-subgroup $P/H_{V\text{-sing}}$. The main difficulty in applying Corollary 2.9 lies in the determination of the smallest positive degree r with non-zero $H^r(H, M)$, and then in the question whether $\inf: H^r(H/H_{V\text{-sing}}, M^{H_{V\text{-sing}}}) \to H^r(H, M)$ is non-zero. To deal with these questions (in particular the first one), we can use the Hochschild-Serre spectral sequence. Indeed, P has order coprime to $|H/P|$ and $H^i(H/P, M^P) = 0$ for $i > 0$, hence

$$H^*(H, M) \cong H^*(P, M)^{H/P} \tag{2.5}$$

(see Evens [21, Exercise 7.2.3]). Thus we have to consider H/P-invariants[1] in $H^*(P, M)$. Since P is a p-group, is it usually quite easy to find non-zero elements in $H^*(P, M)$ of low degree. For example, if $M = R$, there is always a non-zero element in $H^1(P, K)$, which is embedded into $H^1(P, R)$. The following theorem says that under favorable conditions such an element can be changed in such a way to give a non-zero H/P-invariant. In fact we establish maps from the cohomology of P to that of H which are injective on the image of $H^*(P, M^H)$ in $H^*(P, M)$. It is quite extraordinary to have such mappings, as the corestriction certainly does not qualify. In fact, the construction of the mappings in Theorem 2.15 is only possible since we are considering cohomology with values in the polynomial ring, and the mappings are not canonical. The key ingredient is the fact that the g_i constructed in Proposition 1.2 act injectively on the cohomology, as we will see in the proof. It is the antagonism between these g_i and invariants in the image of the transfer (which kill all cohomology) which is crucial to our method. Theorem 2.15 will be used in Section 4 to study the depth of the invariants if (G, V) is visibly flat.

Theorem 2.15. *Let G be a finite group, $N \trianglelefteq G$ a normal Hall subgroup (i.e., the order $|N|$ and the index $[G : N]$ are relatively prime), and $N_0 \trianglelefteq G$ a normal subgroup contained in N. Let $C := \mathcal{C}_G(N)$ be the centralizer. Furthermore, let K be an algebraically closed field of characteristic not dividing $[G : N]$, V and W two finite-dimensional KG-modules, and set $M := W \otimes_K S(V^*)$. Let Δ and Δ_0 be the kernels of the maps $H^*\left(N, M^C\right) \to H^*\left(N, M\right)$ and $H^*\left(N/N_0, M^{C \cdot N_0}\right) \to H^*\left(N/N_0, M^{N_0}\right)$, respectively, induced by containment of modules. Assume that:*

(i) $G/(C \cdot N)$ is abelian, and

(ii) the kernel of the G-action on V^N is contained in $C \cdot N$.

Then with $r := [G : C \cdot N]$ there exists a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Delta_0 & \longrightarrow & H^*\left(N/N_0, M^{C \cdot N_0}\right) & \longrightarrow & \bigoplus_{i=1}^r H^*\left(G/N_0, M^{N_0}\right) \\ & & \big\downarrow & & \big\downarrow \scriptstyle\inf & & \big\downarrow \scriptstyle\inf \\ 0 & \longrightarrow & \Delta & \longrightarrow & H^*\left(N, M^C\right) & \longrightarrow & \bigoplus_{i=1}^r H^*\left(G, M\right) \end{array} \tag{2.6}$$

with exact rows, where all maps are degree-preserving homomorphisms of R^G-modules (i.e., sending i-th cohomology to i-th cohomology).

[1] Notice that at this point our methods are looping back: We find ourselves using invariant theory for calculations in cohomology, which will give answers to questions on invariant theory!

Proof. By the Hochschild-Serre spectral sequence we have

$$H^*(G,M) = H^*(N,M)^{G/N} \quad \text{and} \quad H^*\left(G/N_0, M^{N_0}\right) = H^*\left(N/N_0, M^{N_0}\right)^{G/N}$$

(see (2.5)). We want to obtain the right-hand maps of (2.6) by transforming elements from $H^*\left(N, M^C\right)$ and $H^*\left(N/N_0, M^{C\cdot N_0}\right)$ into G/N-invariants. We first convince ourselves that elements from C act trivially on these cohomology spaces. By Evens [21, p. 35], the G/N-action on $H^*\left(N, M^C\right)$ is given as follows. Let

$$\cdots \xrightarrow{\partial_3} P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \longrightarrow K$$

be a projective resolution of K as a KG-module. Then the P_i are also KN-projective. Now take $\sigma \in G$. If $\alpha \in H^i(N, M^C)$ is given by $\varphi \in \mathrm{Hom}_{KN}(P_i, M^C)$ with $\varphi \circ \partial_{i+1} = 0$, then $\sigma(\alpha)$ is given by $\sigma(\varphi) \in \mathrm{Hom}_{KN}(P_i, M^C)$ sending $x \in P_i$ to $\sigma\left(\varphi(\sigma^{-1}x)\right)$ (which again lies in M^C). Fix a $\sigma \in C$. Then $\left(\sigma(\varphi)\right)(x) = \varphi(\sigma^{-1}x)$. Set $U := \mathrm{im}\,\partial_i \subseteq P_{i-1}$ and define

$$\psi\colon U \to M^C,\ \partial_i(x) \mapsto \varphi(\sigma^{-1}x - x) \quad \text{for} \quad x \in P_i.$$

This is well defined since $\varphi \circ \partial_{i+1} = 0$, and it is a KN-homomorphism since $\sigma^{-1} \in C = \mathcal{C}_G(N)$. Since P_{i-1} is also an injective KN-module (see Alperin [2, Theorem 6.4]), we can extend ψ to get $\psi \in \mathrm{Hom}_{KN}(P_{i-1}, M^C)$. But now

$$\sigma(\varphi) - \varphi = \psi \circ \partial_i$$

is a coboundary, hence $\sigma(\alpha) = \alpha$. The same argument shows that C also acts trivially on $H^*\left(N/N_0, M^{C\cdot N_0}\right)$.

Set $A := G/(C \cdot N)$ and fix a linear character $\chi \in A^* := \mathrm{Hom}(A, K^\times)$. By the canonical epimorphism $G \to A$, χ can be evaluated at any $\sigma \in G$. The next goal is to find a relative invariant $g_\chi \in R := S(V^*)$ with respect to χ which acts injectively on $H^*(N,M)$ and $H^*\left(N/N_0, M^{N_0}\right)$. Let $\hat{N} \trianglelefteq G$ be the kernel of the G-action on V^N, and consider the ideal $I := \mathcal{I}_R(V^N)$ (see (1.2)), which is a G-stable prime ideal. We have $R/I = S\left(V^*/(V^N)^\perp\right)$, and it is easy to see that the kernel of the G-action on $V^*/(V^N)^\perp$ is exactly $\hat{N}$, so this is also the kernel of the action on R/I. Hence $G/\hat{N}$ acts faithfully on R/I, and by Kemper [37, Lemma 2.1] there exists an embedding φ: $K(G/\hat{N}) \hookrightarrow R/I$ of $K(G/\hat{N})$-modules. Let $H \leq G$ be a complement of N (i.e., $G = N \rtimes H$) and set

$$T_\chi := \frac{1}{|H|} \sum_{\sigma \in H} \chi(\sigma^{-1})\sigma \in KG.$$

By the hypothesis (ii) we have $\hat{N} \subseteq C \cdot N$, so $\chi(\sigma)$ only depends on the class $\sigma\hat{N}$. With $\varphi(1) =: f + I$ we get

$$T_\chi(f+I) = \frac{[\hat{N}:N]}{|H|} \sum_{\sigma \in G/\hat{N}} \chi(\sigma^{-1})\sigma(f+I) = \frac{[\hat{N}:N]}{|H|} \cdot \varphi\Big(\sum_{\sigma \in G/\hat{N}} \chi(\sigma^{-1})\sigma \Big) \neq 0.$$

Therefore the image of T_χ (as a mapping on R) is not contained in I, so there also exists a homogeneous $h \in T_\chi(R) \setminus I$. Now take

$$g := \prod_{\tau \in N} \tau(h).$$

Then $g \in R^N \setminus (I \cap R^N)$, and for $\sigma \in H$ we have

$$\sigma(g) = \prod_{\tau \in N} \sigma\tau(h) = \prod_{\tau \in N} \sigma\tau\sigma^{-1}(\chi(\sigma)h) = \chi(\sigma)^{|N|} g.$$

Finally, choose $k \in \mathbb{N}$ such that $k|N| \equiv 1 \mod |A|$ and set $g_\chi := g^k$. Then $\sigma(g_\chi) = \chi(\sigma) g_\chi$ for all $\sigma \in G$, and $g_\chi \in R^N \setminus (I \cap R^N)$. The latter property allows us to apply Proposition 1.2, which yields that multiplication with g_χ has a left inverse in $\mathrm{End}_{KN}(R)$. From this it follows that multiplication with g_χ is injective on $H^*(N,M)$ and $H^*\left(N/N_0, M^{N_0}\right)$, which are $\mathrm{End}_{KN}(R)$-modules.

Using this construction, we get a relative invariant g_χ for every $\chi \in A^*$. Now we construct the maps which make (2.6) commutative. Let η: $H^*\left(N, M^C\right) \to H^*(N,M)$ be the mapping induced from the embedding $M^C \subseteq M$, so $\Delta = \ker\eta$. For $\chi \in A^*$ define

$$\Phi_\chi\colon H^*\left(N, M^C\right) \to H^*(N,M),\ \alpha \mapsto g_{\chi^{-1}}\eta\left(T_\chi(\alpha)\right).$$

Here the application of T_χ comes from the G-action on $H^*\left(N, M^C\right)$. By construction, the image of Φ_χ lies in $H^*(N,M)^{G/N} = H^*(G,M)$. It is also clear that Φ_χ is an R^G-homomorphism. Since η commutes with the G-action (on $H^*\left(N, M^C\right)$ and on $H^*(N,M)$, respectively), $\Delta = \ker(\eta)$ lies in the kernel of Φ_χ. The Φ_χ can be put together to obtain one map Φ: $H^*\left(N, M^C\right) \to \bigoplus_{\chi\in A^*} H^*(G,M)$. This will be the lower right-hand map in (2.6). Assume $\Phi(\alpha) = 0$. Then for all $\chi \in A^*$ we have $\Phi_\chi(\alpha) = 0$, hence $g_{\chi^{-1}}T_\chi\left(\eta(\alpha)\right) = 0$ and therefore $T_\chi\left(\eta(\alpha)\right) = 0$ by the injectivity of $g_{\chi^{-1}}$. But since K is algebraically closed,

$$\sum_{\chi\in A^*} T_\chi = \frac{1}{|H|}\sum_{\sigma\in H}\Big(\sum_{\chi\in A^*}\chi(\sigma^{-1})\Big)\sigma = \frac{|A|}{|H|}\sum_{\substack{\sigma\in H,\\ \sigma N\subseteq C\cdot N}}\sigma.$$

If $\sigma N \subseteq C\cdot N$, then $\sigma(\eta(\alpha)) = \eta(\sigma(\alpha)) = \eta(\alpha)$, hence we obtain

$$\eta(\alpha) = \frac{|A|}{|H|}\sum_{\substack{\sigma\in H,\\ \sigma N\subseteq C\cdot N}}\sigma(\eta(\alpha)) = \sum_{\chi\in A^*} T_\chi\left(\eta(\alpha)\right) = 0,$$

so $\alpha \in \Delta$. This shows the exactness of the lower row in (2.6).

Now let η_0: $H^*\left(N/N_0, M^{C\cdot N_0}\right) \to H^*\left(N/N_0, M^{N_0}\right)$ be the map induced by $M^{C\cdot N_0} \subseteq M^{N_0}$, and for $\chi \in A^*$ define

$$\Psi_\chi\colon H^*\left(N/N_0, M^{C\cdot N_0}\right) \to H^*\left(G/N_0, M^{N_0}\right),\ \alpha \mapsto g_{\chi^{-1}}\eta_0\left(T_\chi(\alpha)\right).$$

As for Φ_χ we obtain that Ψ_χ is an R^G-homomorphism, and that the kernel of the sum Ψ of the Ψ_χ is $\Delta_0 = \ker\eta_0$. Moreover, for each $\chi \in A^*$ the diagram

$$\begin{array}{ccc} H^*\left(N/N_0, M^{C\cdot N_0}\right) & \xrightarrow{\Psi_\chi} & H^*\left(G/N_0, M^{N_0}\right) \\ \big\downarrow \mathrm{inf} & & \big\downarrow \mathrm{inf} \\ H^*\left(N, M^C\right) & \xrightarrow{\Phi_\chi} & H^*(G,M) \end{array}$$

commutes, since the action of G (and thus T_χ) commutes with $\mathrm{inf}\colon H^*\left(N/N_0, M^{C\cdot N_0}\right) \to H^*(N, M^C)$. This completes the proof. $\square$

The following corollary is tailored for applying Corollary 2.9 in the case where H is the normalizer of a Sylow p-subgroup (see before Theorem 2.15). Thus in the following we can think of N as being a Sylow p-subgroup, G its normalizer, and $N_0 = G_{V\text{-sing}}$.

Corollary 2.16. *With the notation and hypotheses of Theorem 2.15, assume that for a non-negative integer i the composition*

$$H^i\left(N/N_0, M^{C\cdot N_0}\right) \longrightarrow H^i\left(N/N_0, M^{N_0}\right) \xrightarrow{\mathrm{inf}} H^i(N,M)$$

is not the zero-map. Then also

$$H^i\left(G/N_0, M^{N_0}\right) \xrightarrow{\mathrm{inf}} H^i(G,M)$$

is non-zero.

Proof. This follows by a simple diagram chase, comparing the diagram (2.6) with

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Delta_0 & \longrightarrow & H^*\left(N/N_0, M^{C\cdot N_0}\right) & \longrightarrow & H^*\left(N/N_0, M^{N_0}\right) \\ & & \downarrow & & \downarrow \text{inf} & & \downarrow \text{inf} \\ 0 & \longrightarrow & \Delta & \longrightarrow & H^*\left(N, M^C\right) & \longrightarrow & H^*\left(N, M\right), \end{array}$$

which is also commutative with exact rows. □

3 Groups of order not divisible by p^2

The most favorable situation for the application of Corollary 2.9 is if there exists a strongly p-embedded subgroup $H \le G$ such that $H_{V\text{-sing}} = \{1\}$. This is true irrespective of the choice of V if the order of G is divisible by p but not by p^2. Indeed, let $P \le G$ be a Sylow p-subgroup and $H = \mathcal{N}_G(P)$ the normalizer. Then H is strongly p-embedded in G since P is cyclic of order p. Moreover, the subgroup H_p in H generated by all elements of order a power of p coincides with P, and the only element σ of P with $V^\sigma \supsetneqq V^P$ is $\sigma = 1$. Therefore Corollary 2.9 applies, and we only have to determine the smallest positive r with $H^r(G, M) \ne 0$, or, equivalently, $H^r(H, M) \ne 0$, where $M = S(V^*) \otimes_K W$ with another KG-module W. Let us state this result for future reference.

Theorem 3.1. *Let G be a finite group and K a field of characteristic p such that $|G|$ is divisible by p but not by p^2. For a finite-dimensional G-module V, let r be the smallest positive integer such that $H^r(G, R) \ne 0$, where $R = S(V^*)$. Then*

$$\operatorname{depth}(R^G) = \min\{\dim_K(V^P) + r + 1, \dim_K(V)\}$$

with $P \le G$ a Sylow p-subgroup. In particular, (G, V) is flat (as defined on page 3) if and only if $r = 1$ or $\dim_K(V^P) \ge \dim_K(V) - 2$.

By (2.5) we have to calculate the H/P-invariants in $H^*(P, M)$. Since $P = \langle\sigma\rangle \cong Z_p$ is cyclic, the spaces $H^i(P, M)$ can be represented as

$$\begin{array}{ll} H^i(P, M) = M^\sigma/(\sigma - 1)^{p-1}M & \text{for } i > 0 \text{ even}, \\ H^i(P, M) = \ker(\sigma - 1)^{p-1}/(\sigma - 1)M & \text{for } i \text{ odd}, \end{array} \tag{3.1}$$

(see Evens [21, p. 6]), where we used the polynomial identity $1 + X + \cdots + X^{p-1} = (X - 1)^{p-1}$ over K to express the transfer as $(\sigma - 1)^{p-1}$. We will explain at the beginning of Section 5.1 how the $H^i(P, R)$ and $H^i(H, R)$ can be evaluated without having to compute the entire invariant ring. On the other hand, in Section 3.1 we will concentrate on permutation groups, where we can use the Eckmann-Shapiro lemma (see Evens [21, Proposition 4.1.3]) to compute cohomology.

For any practical purposes we have to determine the action of H/P on $H^*(P, M)$. More generally, H/P acts on $H^*(P, U)$ for any KH-module U. This action arises naturally from taking a projective resolution of K over KH and considering it as a resolution over KP (see Evens [21, p. 35]). However, the formulas (3.1) (which hold for any KH-module M) were obtained from a much simpler, periodic resolution over KP. Therefore one has to transform the action from the one resolution to the other, which is a rather cumbersome task. For the first and second cohomology, the result is as follows: Let α be an element of $H^1(P, U)$ or $H^2(P, U)$. By (3.1), α can be represented by a $u \in U$ with special properties, depending on whether α lies in the first or second cohomology. It turns out that in either case the application $\tau\alpha$ can be represented by

$$\chi(\tau) \cdot \tau u,$$

where the linear character χ: $H \to \mathbb{F}_p$ is given by

$$\tau^{-1}\rho\tau = \rho^{\chi(\tau)} \tag{3.2}$$

for $\tau \in H$ and $\rho \in P$. It is enough to know the action of H/P on the first and second cohomology. Indeed, $H^*(P,U)$ is a module over the cohomology ring $H^*(P,K)$ by the cup product, and the H/P-action respects the ring structure of $H^*(P,K)$ and the module structure of $H^*(P,U)$. But Benson [4, Corollary 3.5.4] implies that there exists an element $y \in H^2(P,K)$ such that $H^*(P,U)$ is generated by $H^1(P,U)$ and $H^2(P,U)$ as a module over the (polynomial) subalgebra $K[y] \subset H^*(P,K)$. Thus we obtain the following rule.

Proposition 3.2. *Let H be a finite group with a normal subgroup $P \trianglelefteq H$ of order p, and let K be a field of characteristic p. For a KH-module U and $i > 0$, let $\alpha \in H^i(P,U)$ be represented by an $u \in U$ according to (3.1). Then by the action of H/P on $H^*(P,U)$, a $\tau \in H/P$ sends α to the element represented by*

$$\chi(\tau)^{[(i+1)/2]} \cdot \tau(u),$$

where the square bracket denotes the greatest integer function, and $\chi\colon H \to \mathbb{F}_p$ is given by (3.2).

3.1 Permutation groups

Now we consider the case of permutation actions. So let $R = K[x_1,\ldots,x_n]$ be a polynomial ring over a field K of positive characteristic p, and let G be a finite group acting on R by permutations of the indeterminates x_i, such that p^2 does not divide the group order $|G|$. Choose a Sylow p-subgroup $P \le G$ and let $N := \mathcal{N}_G(P)$ be the normalizer. The set $\{x_1,\ldots,x_n\}$ decomposes into a disjoint union of P-orbits $X_1,\ldots,X_k$. Let $H \le N$ be the subgroup of all elements $\sigma \in N$ with $\sigma X_i \subseteq X_i$ for all i. Then $P \trianglelefteq H$, and H acts on P with the centralizer $\mathcal{C}_H(P)$ as the kernel of the action. Set

$$m := |H/\mathcal{C}_H(P)|,$$

which is a divisor of $p-1$. Although the procedure to obtain the order m is a bit lengthy to describe, it is clear that m and also the number k of P-orbits can easily be determined for any given permutation group.

Theorem 3.3. *In the above situation we have*

$$\mathrm{depth}(R^G) = \min\{2m+k, n\}.$$

Proof. If $p \nmid |G|$, then $P = \{1\}$ and $k = n$, so the theorem is correct by Hochster and Eagon [28]. Therefore we can assume that p divides $|G|$. G acts linearly on $V = K^n$ by permutations of the standard basis vectors, and $R \cong S(V^*)$ as K-algebras and as KG-modules, since V is self-dual. By Theorem 3.1 we have $\mathrm{depth}(R^G) = \min\left\{\dim_K(V^P) + r + 1, n\right\}$, where r is the smallest positive number such that $H^r(N,R) \ne 0$. Clearly $\dim_K(V^P) = k$, so we have to show that $r = 2m-1$.

Since N permutes the x_j, it also acts as a permutation group on the set of all monomials in the x_j. Therefore R decomposes into a direct sum of NG-submodules of the form

$$U_t = \sum_{\sigma \in N} K\sigma(t)$$

with t a monomial. For an $i > 0$, we have $H^i(N,R) \ne 0$ if and only if there exists a monomial t such that $H^i(N,U_t) \ne 0$. But the module U_t can be obtained by taking the trivial module of the stabilizer $N_t := \{\sigma \in N \mid \sigma(t) = t\}$ and inducing up to N: $U_t = \mathrm{ind}_{N_t}^N K$. Now the Eckmann-Shapiro lemma (see Evens [21, Proposition 4.1.3]) yields

$$H^i(N,U_t) \cong H^i(N_t,K).$$

Let us assume that this is non-zero. Then $P \subseteq N_t$ (since otherwise $p \nmid |N_t|$), and therefore $H \subseteq N_t$ by the definition of H. With (2.5) we obtain

$$H^i(N,U_t) \cong H^i(P,K)^{N_t/P} \subseteq H^i(P,K)^{H/P}.$$

Therefore $H^i(N,R) \neq 0$ implies $H^i(P,K)^{H/P} \neq 0$. Conversely, assume that $H^i(P,K)^{H/P} \neq 0$. Consider the monomial

$$t_P := \prod_{j=1}^{k} \prod_{x_l \in X_j} x_l^j.$$

By the definition of H, $N_{t_P} = H$, and hence

$$H^i(N, U_{t_P}) \cong H^i(P,K)^{H/P} \neq 0,$$

so we obtain $H^i(N,R) \neq 0$. Thus the smallest positive i with $H^i(N,R) \neq 0$ is equal to the smallest positive i with $H^i(P,K)^{H/P} \neq 0$. H/P acts trivially on K, and it acts on P as the group $H/(H \cap \mathcal{C}_G(P))$, which is cyclic of order m. Hence by Proposition 3.2 there is an H/P-invariant in $H^i(P,K)$ if and only if m divides $[(i+1)/2]$. The smallest such $i > 0$ is $2m-1$, so $r = 2m-1$, which was to be shown. □

From Theorem 3.3 we see that (G,V) is flat if and only if $2m + k \geq n$ or $m = 1$, which is relatively rare. In [37] the author classified the finite groups G such that the invariant ring of G with respect to the regular representation V_{reg} is Cohen-Macaulay (see [37, Theorem 2.7]). By using Theorem 3.3, we can go further and in the case of groups of order not divisible by p^2 determine the depth of the invariant ring.

Corollary 3.4. *Let G be a finite group and K a field of positive characteristic p, such that the group order $|G|$ is a multiple of p but not of p^2. Then if V_{reg} is the regular KG-module, we have*

$$\operatorname{depth}\left(S(V_{\text{reg}})^G\right) = \frac{|G|}{p} + 2,$$

except in the case $|G| = 2$, where the depth is 2. In particular, (G, V_{reg}) is flat.

Proof. We have to consider the regular action of G on itself and determine the numbers k and m occurring in Theorem 3.3. First, the number k of P-orbits for P a Sylow p-subgroup is $|G|/p$, since the orbits are exactly the right cosets of P. Moreover, the subgroup $H \leq \mathcal{N}_G(P)$ stabilizing all P-orbits clearly is P itself, hence $m = 1$. Thus Theorem 3.3 yields

$$\operatorname{depth}\left(S(V_{\text{reg}})^G\right) = \min\left\{|G|/p + 2, |G|\right\}.$$

□

Let us now consider vector invariants of permutation representations. In other words, given a permutation action on indeterminates $x_1, \ldots, x_n$, we take new indeterminates $x_{i,j}$ with $1 \leq i \leq r$ and $1 \leq j \leq n$, on which G acts by the second index. Vector invariants (usually of linear representations) have been a classical area of interest in invariant theory, and have recently enjoyed some interest in modular invariant theory (Richman [48], Campbell and Hughes [11], Campbell et al. [13]). Vector invariants of permutation groups have been studied by Fleischmann [23] and Kemper [34]. We make an interesting observation here: The number m occurring in Theorem 3.3 does not change as one passes to vector invariants, while the number k gets multiplied by r. As an (important) example, we evaluate the depth of vector invariants of the symmetric group S_n on n symbols where $p \leq n < 2p$.

Corollary 3.5. *Let $R = K[x_{1,1}, \ldots, x_{r,n}]$ be a polynomial ring in rn indeterminates over a field K of characteristic p such that $p \leq n < 2p$. Consider the action of the symmetric group $G = S_n$ on the $x_{i,j}$ by the second index. Then*

$$\operatorname{depth}(R^G) = rn - (p-1) \cdot \max\{r-2, 0\}.$$

In particular, R^G is Cohen-Macaulay if and only if $r \leq 2$.

Proof. It is elementary to see that the number m occurring in Theorem 3.3 is $p-1$, and the number k is $r(n-p+1)$. Now the result follows from Theorem 3.3. □

As Corollary 3.5 shows, vector invariants of the symmetric group are badly behaved with respect to the Cohen-Macaulay property, even though the invariant ring for just one copy of the variables is a polynomial ring. This is a general phenomenon. In fact, the author proved in [37] that for any group G with $p \mid |G|$ and for any faithful KG-module V the ring of vector invariant of sufficiently many copies of V is not Cohen-Macaulay. It has been conjectured by Ian Hughes and others that three copies will always suffice. We can confirm this conjecture for permutation groups of order not divisible by p^2:

Corollary 3.6. *Let $G \leq S_n$ be a permutation group of order divisible by p but not by p^2. If G acts as above on the polynomial ring $R = K[x_{1,1}, \ldots, x_{r,n}]$ in rn indeterminates over a field K of characteristic p, then R^G is not Cohen-Macaulay if $r \geq 3$.*

Proof. Take the numbers m and k from Theorem 3.3 for the action on one set of variables $x_1, \ldots, x_n$. As we have seen, passing to vector invariants leaves m fixed but multiplies k by r. By Theorem 3.3, the ring R^G of vector invariants is Cohen-Macaulay if and only if

$$2m + rk \geq rn,$$

which is equivalent to $r \leq 2m/(n-k)$. But in the definition of m before Theorem 3.3, the group $H/H \cap \mathcal{C}_G(P)$ is embedded into the automorphism group of P, hence its order m is at most $p-1$. Moreover, the Sylow p-subgroup P must have at least one orbit of length p, therefore $n-k \geq p-1$. Thus we obtain that if R^G is Cohen-Macaulay, then

$$r \leq \frac{2m}{n-k} \leq \frac{2(p-1)}{p-1} = 2,$$

which proves the corollary. □

3.2 Friends of permutation modules

It is also interesting to look at some KG-modules which are derived from permutation modules. Assume that G acts as a faithful transitive permutation group on a basis $e_1, \ldots, e_n$ of a vector space W over a field K whose characteristic p divides n. Then we consider the KG-modules

$$V := \{\alpha_1 e_1 + \cdots + \alpha_n e_n \in W \mid \sum_{i=1}^{n} \alpha_i = 0\} \subset W \quad \text{and} \quad U := V/K \cdot (e_1 + \cdots + e_n)\,.$$

In [37, Corollary 2.8] the author showed that under relatively mild conditions the invariant rings $S(V)^G$ and $S(U)^G$ are not Cohen-Macaulay. These conditions were further weakened in Kemper [36, Proposition 3.16], where it was shown that $S(V)^G$ is not Cohen-Macaulay if $p \geq 5$, and $S(U)^G$ is not Cohen-Macaulay if in addition $n > 5$. Under the assumption that p^2 does not divide $|G|$, we can now get exact statements about the depth of the invariant rings.

Proposition 3.7. *With the above notation assume that $p^2 \nmid |G|$. Then (G,V) is flat and (G,U) is flat for $p > 2$, and we have:*

(a) If $n \geq 5$ then

$$\operatorname{depth}\left(S(V)^G\right) = \frac{n}{p} + 2.$$

(b) If $n \geq 5$ and $p \geq 3$ then

$$\operatorname{depth}\left(S(U)^G\right) = \frac{n}{p} + 2.$$

(c) If $n < 5$, then $S(V)^G$ and $S(U)^G$ are Cohen-Macaulay.

Proof. Since G acts transitively, the point-stabilizer G_{e_i} of every e_i has index n in G. Since p divides n, it also divides the order of G, and a Sylow p-subgroup P fixes none of the e_i, since otherwise p^2 would divide $|G|$. From Kemper [36, Proposition 3.16] (or rather its proof) we see that the first cohomology $H^1(G,V)$ and $H^1(G,U)$ is non-zero (assuming $p \neq 2$ for $H^1(G,U)$). Since U and V are direct summands of $S(U)$ and $S(V)$, respectively, we also have that the first cohomologies with values in the polynomial rings are non-zero. The flatness follows by Theorem 3.1. To obtain formulas for the depth we have to determine the dimensions of the fixed spaces $(V^*)^P$ and $(U^*)^P$.

It is easy to see that as a KG-module the dual V^* is isomorphic to

$$\overline{W} := W / K \cdot (e_1 + \cdots + e_n),$$

and U is self-dual. An element of $\overline{W}$ represented by $v := \alpha_1 e_1 + \cdots + \alpha_n e_n \in W$ is P-invariant if and only if $(\sigma - 1)v = \alpha(e_1 + \cdots + e_n)$ with $\alpha \in K$, where σ is a generator of P. Then $\alpha = \alpha_1 - \alpha_{\sigma(1)}$. If $e_{i_1}, \ldots, e_{i_m}$ are representatives of the P-orbits, then $\sum_{k=1}^m \sum_{j=1}^p j \cdot \sigma^j(e_{i_k})$ represents a P-invariant in $\overline{W}$ with $\alpha \neq 0$. If $p \neq 2$, this invariant lies in U. It is now easy to see that

$$\overline{W}^P \to K^m, \ \alpha_1 e_1 + \cdots + \alpha_n e_n + K(e_1 + \cdots + e_n) \mapsto (\alpha_{i_1} - \alpha_{i_m}, \ldots, \alpha_{i_{m-1}} - \alpha_{i_m}, \alpha_1 - \alpha_{\sigma(1)})$$

is an isomorphism of vector spaces, and $\overline{W}^P = U^P$. Hence Theorem 3.1 yields

$$\operatorname{depth}\left(S(V)^G\right) = \min\{2 + m, n - 1\} \quad \text{and} \quad \operatorname{depth}\left(S(U)^G\right) = \min\{2 + m, n - 2\},$$

where $m = n/p$. If $n \geq 5$, we have $2 + m \leq n - 1$, and $2 + m \leq n - 2$ for $p \neq 2$. Thus (a) and (b) follow. On the other hand, if $n < 5$ then $n - 1 \leq 2 + m$, which proves (c). □

Example 3.8. We apply Proposition 3.7 to the symmetric group $G = S_p$. G acts as a reflection group on both U and V. On U, it acts irreducibly, and the action on V comes from an action in characteristic zero by reduction modulo p. In both cases the proposition says that the depth of the invariant ring is exactly 3. We therefore have modular reflection groups such that the homological dimensions of the invariant rings become arbitrarily large. It is all the more surprising that by Kemper and Malle [38] the invariant fields $K(U)^{S_p}$, i.e., the fields of fractions of the invariant rings, are purely transcendental over K for all p. ◁

4 Visible flatness

We will define visible flatness by a number of properties which form the hypotheses of the main Theorem 4.2 of this section. We first prove the theorem and some remarks about the properties constituting visible flatness, and then present applications.

4.1 The theorem and its hypotheses

We start with the following definition.

Definition 4.1. *Let G be a finite group, K a field of positive characteristic p, and V a finite-dimensional KG-module. Let $P \leq G$ be a Sylow p-subgroup, $H := \mathcal{N}_G(P)$ its normalizer, and $C := \mathcal{C}_G(P)$ the centralizer. Then (G,V) is called* **visibly flat** *if the following properties hold:*

(P_1) P is a trivial intersection subgroup, i.e., ${}^\sigma P \cap P \in \{P, \{1\}\}$ for all $\sigma \in G$,

(P_2) $H/(C \cdot P)$ is abelian,

(P_3) P is not *generated by the subset $\left\{\sigma \in P \mid V^\sigma \supsetneqq V^P\right\}$, and*

(P_4) *the kernel of the H-action on V^P lies in $C \cdot P$.*

By bringing Corollaries 2.9 and 2.16 together, we can now prove that visible flatness implies flatness. We formulate the theorem for the module M^G of relative invariants with respect to a linear character. This module is the invariant ring if the character is trivial. Notice that $\dim(M^G)$ is equal to $\dim_K(V)$ unless $M^G = 0$.

Theorem 4.2. *Let G be a finite group, K a field of positive characteristic p, and V, W finite-dimensional KG-modules with $\dim_K(W) = 1$, and assume that (G, V) is visibly flat. Then with $R := S(V^*)$ and $M := W \otimes_K R$ we have*

$$\operatorname{depth}\left(M^G\right) = \min\left\{\dim_K(V^P) + 2, \dim_K(M^G)\right\}$$

with $P \le G$ a Sylow p-subgroup. In particular, (G, V) is flat (as defined on page 3).

Proof. (P_1) guarantees that H is a strongly p-embedded subgroup of G, and (P_3) means that $H_{V\text{-sing}}$ is a proper subgroup of P. As a KP-module, W is trivial, therefore M has K as a direct summand. But $P/H_{V\text{-sing}}$ is a non-trivial p-group, hence $H^1(P/H_{V\text{-sing}}, K) \ne 0$. By the directness of K in M, a non-zero element α is sent to a non-zero $\beta \in H^1\left(P/H_{V\text{-sing}}, M^{H_{V\text{-sing}}}\right)$ by the mapping induced by containment of the modules. By Remark 2.8(b) the inflation map into $H^1(P, M)$ sends β to a non-zero element again. Therefore the composition map

$$H^1\left(P/H_{V\text{-sing}}, M^{C \cdot H_{V\text{-sing}}}\right) \to H^1\left(P/H_{V\text{-sing}}, M^{H_{V\text{-sing}}}\right) \to H^1(P, M)$$

is non-zero. As in the proof of Theorem 2.6 we can assume that K is algebraically closed. Now the properties (P_2) and (P_4) make Corollary 2.16 applicable, which yields that the inflation map

$$H^1\left(H/H_{V\text{-sing}}, M^{H_{V\text{-sing}}}\right) \to H^1(H, M)$$

is also non-zero. Now the result follows from Corollary 2.9. □

A great advantage of Theorem 4.2 is that the Properties (P_1)–(P_4) are straight-forward to check without computing any invariants or cohomology. Notice also that (P_1)–(P_4) are independent of the choice of the Sylow p-subgroup P. By looking at (P_1) and (P_2) one already guesses that some classical groups like $\mathrm{SL}_2(q)$ should have interesting representations V that give a visibly flat pair. More generally, the appendix of this paper contains a classification of the pairs (G, V) of a finite group with BN-pair and a KG-module V in defining characteristic, such that (G, V) is visibly flat (Theorem A.8). In the following proposition we will make a number of observations on the Properties (P_1)–(P_4), and how they pass down to subgroups or carry over to different representations. It is convenient to introduce a stronger version of (P_3):

(P_3^+) $V^\sigma = V^P$ for all $\sigma \in P \setminus \{1\}$.

In the following we will say that $P_i(G, V)$ holds to mean that the Property (P_i) holds for the pair (G, V).

Proposition 4.3. *Let G be a finite group with a non-trivial Sylow p-subgroup P, K a field of positive characteristic p, and V, W two finite-dimensional KG-modules. Then we have:*

(a) If G is abelian and P is cyclic, then (G, V) is visibly flat.

(b) If $|P| = p$, then $P_1(G, V)$, $P_2(G, V)$, and $P_3^+(G, V)$ hold.

(c) Let $H \le G$ be a subgroup whose order is divisible by p, and assume that P is abelian. Then $P_1(G, V)$ implies $P_1(H, V)$. Moreover, if $P_1(G, V)$ holds, then each of $P_2(G, V)$, $P_3^+(G, V)$, and $P_4(G, V)$ implies $P_2(H, V)$, $P_3^+(H, V)$, and $P_4(H, V)$, respectively.

(d) Let $n_1, \dots, n_k$ be non-negative integers such that with $m := [\mathcal{N}_G(P) : \mathcal{C}_G(P) \cdot P]$ we have

$$\gcd\{m, n_1, \dots, n_k\} = 1.$$

Then $P_4(G, V)$ implies $P_4(G, S^{n_1}(V) \oplus \cdots \oplus S^{n_k}(V))$.

(e) If $P_3^+(G, V)$ and $P_3^+(G, W)$ hold, then $P_3^+(G, V \oplus W)$ also holds.

Proof. If G is abelian, (P_1), (P_2) and (P_4) follow. Moreover, (P_3) follows if P is cyclic. This proves (a).

In the situation of (b), (P_2) follows since $H/(C \cdot P)$ is a quotient of a subgroup of $\mathrm{Aut}(P)$, and (P_1) and (P_3^+) are clear.

Now let $H \le G$ be a subgroup. $P \le G$ can be chosen in such a way that $H \cap P$ is a Sylow p-subgroup of H. Assume $P_1(G, V)$ holds, and take $\sigma \in H \setminus \mathcal{N}_H(H \cap P)$. Then $\sigma\tau\sigma^{-1} \notin P$ for some $\tau \in H \cap P$, hence $\sigma \notin \mathcal{N}_G(P)$. By $P_1(G, V)$, $P \cap {}^{\sigma}P = \{1\}$ follows, and then also

$$(H \cap P) \cap {}^{\sigma}(H \cap P) = \{1\},$$

which is $P_1(H, V)$. Assume from now on that $P_1(G, V)$ holds, and take $\sigma \in H \setminus \mathcal{N}_G(P)$. Then $\sigma\tau\sigma^{-1} \notin P$ for every $\tau \in P \setminus \{1\}$. Since p divides H, there exists a $\tau \in H \cap P \setminus \{1\}$, and it follows that $\sigma \notin \mathcal{N}_H(H \cap P)$. Therefore

$$\mathcal{N}_H(H \cap P) = H \cap \mathcal{N}_G(P). \tag{4.1}$$

If P is abelian, then $\mathcal{C}_G(P) \cdot P = \mathcal{C}_G(P)$. Hence $P_2(G, V)$ means that $\mathcal{N}_G(P)$ acts as an abelian group on P. But then the subgroup $\mathcal{N}_H(H \cap P)$ also acts as an abelian group on $H \cap P$, so we have $P_2(H, V)$. Furthermore, $P_3^+(G, V)$ implies that $V^P = V^{H \cap P}$, since there exist $\tau \in H \cap P \setminus \{1\}$. Now again by $P_3^+(G, V)$, the only $\sigma \in H \cap P$ for which $V^\sigma \supsetneqq V^{H \cap P}$ can hold is $\sigma = 1$, which is $P_3^+(H, V)$. To conclude $P_4(H, V)$ from $P_4(G, V)$, take a $\sigma \in \mathcal{N}_H(H \cap P)$ which acts trivially on $V^{H \cap P}$. Then σ also acts trivially on the (smaller) space V^P and $\sigma \in \mathcal{N}_G(P)$ by (4.1). Hence by $P_4(G, V)$ we have $\sigma \in \mathcal{C}_G(P) \cdot P = \mathcal{C}_G(P)$ (again using the abelianness of P). A fortiori σ also commutes with all elements from $H \cap P$, so $\sigma \in \mathcal{C}_H(H \cap P)$. This gives $P_4(H, V)$, and completes the proof of (c).

To prove (d), assume that $\sigma \in \mathcal{N}_G(P)$ acts trivially on $U := \left(S^{n_1}(V) \oplus \cdots \oplus S^{n_k}(V)\right)^P$. Since P lies in the kernel of the $\mathcal{N}_G(P)$-action on V^P, we can diagonalize σ on V^P (assuming without loss that K is large enough). If λ is an eigenvalue with eigenvector v, then for $v^{n_i} \in S^{n_i}(V)^P \subseteq U$ we have $\sigma(v^{n_i}) = \lambda^{n_i} v^{n_i}$. Since σ acts trivially on U, it follows that $\lambda^{n_i} = 1$ for all i. By the assumption of (d) there exist integers $l, l_1, \dots, l_k$ such that $lm + l_1 n_1 + \cdots + l_k n_k = 1$. Therefore

$$\sigma^{1-lm}(v) = \lambda^{l_1 n_1 + \cdots + l_k n_k} v = v.$$

Since this holds for any eigenvector v, σ^{1-lm} acts trivially on V^P. If $P_4(G, V)$ holds, then $\sigma^{1-lm} \in \mathcal{C}_G(P) \cdot P$ follows, but from the definition of m we also have $\sigma^{lm} \in \mathcal{C}_G(P) \cdot P$, hence $\sigma \in \mathcal{C}_G(P) \cdot P$. This is $P_4(G, S^{n_1}(V) \oplus \cdots \oplus S^{n_k}(V))$, so (d) is shown.

The statement (e) is clear. ∎

In the following we will apply Theorem 4.2 to groups with a BN-pair of rank one and their subgroups. We will formulate the results only for the invariant rings R^G and not for the modules of relative invariants, since relative invariants always require the distinction whether $M^G = 0$ or not. It is clear that all results can be obtained in the setting of relative invariants as well.

4.2 Subgroups of $\mathbf{SL}_2(q)$

The following theorem contains the main result (Theorem 4.3) of Shank and Wehlau [49] as a special case.

Theorem 4.4. *Let G be a subgroup of $\mathrm{SL}_2(q)$ such that p divides $|G|$, where $q = p^r$, and let U be the natural representation of $\mathrm{SL}_2(q)$ over $K \supseteq \mathbb{F}_q$. Set $m := [\mathcal{N}_G(P) : \mathcal{C}_G(P)]$, where $P \le G$ is a Sylow p-subgroup, and let $n_1, \ldots, n_k$ be integers such that $0 \le n_i < p$ and*

$$\gcd\{m, n_1, \ldots, n_k\} = 1.$$

Set

$$V := S^{n_1}(U) \oplus \cdots \oplus S^{n_k}(U)$$

(with S^i the i-th symmetric power) and $R := S(V^*)$. Then (G, V) is flat, and we have

$$\operatorname{depth}\left(R^G\right) = k + \min\{2, n_1 + \cdots + n_k\}.$$

Moreover, if $q = p$, then the restriction $n_i < p$ can be lifted, and we have the more general formula

$$\operatorname{depth}\left(R^G\right) = k + \min\left\{2 + \sum_{i=1}^k \left[\frac{n_i}{p}\right], n_1 + \cdots + n_k\right\},$$

where $[n_i/p]$ is the greatest integer $\le n_i/p$.

Proof. Set $\widetilde{G} := \mathrm{SL}_2(q)$. By Theorem A.8, $(\widetilde{G}, U)$ is visibly flat. From Proposition 4.3(c) we get the properties $P_1(G, U)$, $P_2(G, U)$, and $P_4(G, U)$. The first two properties are purely group-theoretic, and we have $P_4(G, V)$ by Proposition 4.3(d). To see that (G, V) is visibly flat, we only have to show that $P_3^+(\widetilde{G}, V)$ holds and apply Proposition 4.3(c) again. In the case $K = \mathbb{F}_p$ we have $|P| = p$, so $P_3^+(\widetilde{G}, V)$ follows (irrespective of the representation) by Proposition 4.3(b). For more general K, in order to see $P_3^+(\widetilde{G}, S^n(U))$ for $n < p$, we must compare the dimensions of $S^n(U)^\tau$ and $S^n(U)^P$ for $1 \ne \tau \in P$. These dimensions are encoded in the Hilbert series

$$H(S(U)^\tau, t) := \sum_{n=0}^{\infty} \dim_K \left(S^n(U)^\tau\right) t^n,$$

and correspondingly for $S(U)^P$. Since τ has the Jordan canonical form $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, the Hilbert series is known to be

$$H(S(U)^\tau, t) = \frac{1}{(1-t)(1-t^p)} \tag{4.2}$$

(see, Smith [50, Example 5.6.3]). Hence for $n < p$ we have

$$1 = \dim_K \left(S^n(U)^\tau\right) \ge \dim_K \left(S^n(U)^P\right) \ge 1, \tag{4.3}$$

and equality follows. Thus $P_3^+(\widetilde{G}, S^{n_i}(U))$ holds for all i, and $P_3^+(\widetilde{G}, V)$ follows by Proposition 4.3(e).

We have shown that (G, V) is visibly flat, hence Theorem 4.2 yields

$$\operatorname{depth}\left(R^G\right) = \min\left\{\dim_K(V^P) + 2, \dim_K(V)\right\}.$$

Now by Equation (4.3) we have $\dim_K\left(S^n(U)^P\right) = 1$ for $n < p$. If $|P| = p$, then (4.2) gives the Hilbert series of $S(U)^P$, and we obtain $\dim_K\left(S^n(U)^P\right) = 1 + [n/p]$. From this the result follows. □

Remark 4.5. The number m in Theorem 4.4 is a divisor of $(q-1)/2$ if $p \ne 2$, and of $q-1$ if $p = 2$. This follows from the formula

$$\begin{pmatrix} a & y \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & y \\ 0 & a^{-1} \end{pmatrix}^{-1} = \begin{pmatrix} 1 & a^2 x \\ 0 & 1 \end{pmatrix}, \tag{4.4}$$

which implies that the $((q-1)/2)$-th (or $(q-1)$-st if $p = 2$) power of any $\sigma \in H = \mathcal{N}_G(P)$ lies in $C = \mathcal{C}_G(P)$, and H/C is cyclic. ◁

Example 4.6. For $G = \mathrm{SL}_2(q)$, the assumption of Theorem 4.4 on the $n_i < p$ reads

$$\gcd\{q-1, n_1, \ldots, n_k\} \leq 2.$$

In this case we get $\operatorname{depth}(R^G) = \min\{2+k, \dim_K(V)\}$. If $k = 1$ and $n_1 \geq 2$ then the depth is always three. Notice that in the case $k = 1$, i.e., $V = S^n(U)$, we are looking at the action of $\mathrm{SL}_2(q)$ on binary forms of degree n, which is a classical field of interest in invariant theory (see Hilbert [27]).

To see that the above condition on the greatest common divisor cannot be omitted, consider the example $G = \mathrm{SL}_2(\mathbb{F}_p)$ with $p > 2$ and $V = S^{p-1}(\mathbb{F}_p^2)$, which is p-dimensional. The center C of G acts trivially on V, so we only need to consider the quotient G/C. The normalizer $H = \mathcal{N}_{G/C}(P)$ of a Sylow p-subgroup is isomorphic to $Z_p \rtimes Z_{(p-1)/2}$, where Z_p is identified with $\mathbb{F}_p$ and $Z_{(p-1)/2}$ is the subgroup of squares in the multiplicative group $\mathbb{F}_p^\times$, which acts on $\mathbb{F}_p$ by multiplication (see (4.4)). It is easily seen that V is projective and indecomposable as a KZ_p-module and therefore also as a KH-module. Since $V^H \neq 0$, V is the (uniquely determined) projective indecomposable module containing the trivial module. But then V is the regular KZ_p-module with $Z_{(p-1)/2}$ also acting by permutations. Since the action of $Z_{(p-1)/2}$ comes from multiplication on $\mathbb{F}_p$ with squares, it follows that H is the normalizer of Z_p in the alternating group A_p. In particular, H is strongly p-embedded into A_p, so

$$\operatorname{depth}\left(S(V^*)^H\right) = \operatorname{depth}\left(S(V^*)^{A_p}\right)$$

by Kemper [37, Corollary 1.2]. But $S(V^*)^{A_p}$ is a hypersurface (see Smith [50, Corollary 1.3.2]), hence $\operatorname{depth}\left(S(V^*)^H\right) = p$, and $S(V^*)^H$ is Cohen-Macaulay. Since p does not divide the index $[G/C : H]$, it follows by Campbell et al. [12] that $S(V^*)^G$ is also Cohen-Macaulay, so

$$\operatorname{depth}\left(S(V^*)^G\right) = p.$$

If $p \geq 5$, this is greater than 3, so omitting the hypothesis on the greatest common divisor in Theorem 4.4 would indeed make it false. ◁

Example 4.7. We can also consider the special unitary group

$$G = \mathrm{SU}_2(q),$$

which is embedded into $\mathrm{SL}_2(q^2)$. With a suitable choice of a Hermitian form, the Sylow p-subgroup becomes upper unipotent, and its normalizer is

$$\mathcal{N}_G(P) = \left\{ \begin{pmatrix} a & x \\ 0 & a^{-1} \end{pmatrix} \mid a \in \mathbb{F}_{q^2}, a^{q-1} = 1 \right\}.$$

Therefore Theorem 4.4 applies to the action of G on $V = S^{n_1}(U) \oplus \cdots \oplus S^{n_k}(U)$ (with $U = \mathbb{F}_{q^2}^2$ the natural module) if $0 \leq n_i < p$ and

$$\gcd\{q-1, n_1, \ldots, n_k\} \leq 2.$$

◁

4.3 Subgroups of $\mathrm{SO}_3(q)$ and $\mathrm{SU}_3(q)$

The special orthogonal group $\mathrm{SO}_3(q)$ is a further example of a (non-perfect) finite group with a BN-pair of rank 1 (since $\mathrm{SO}_3(q) \cong \mathrm{PGL}_2(q)$, see Huppert [31, Satz 10.11]). Very much along the lines of Theorem 4.4, we obtain the following result.

Theorem 4.8. *Let G be a subgroup of $\mathrm{SO}_3(q)$ such that p divides $|G|$, where $q = p^r$, and let U be the natural three-dimensional $K(\mathrm{SO}_3(q))$-module with $\mathbb{F}_q \subseteq K$. Set $m := [\mathcal{N}_G(P) : \mathcal{C}_G(P)]$, where $P \leq G$ is a Sylow p-subgroup. Let $n_1, \ldots, n_k$ be integers such that $0 \leq n_i < p$ and*

$$\gcd\{m, n_1, \ldots, n_k\} = 1,$$

and set

$$V := S^{n_1}(U) \oplus \cdots \oplus S^{n_k}(U).$$

Then (G, V) *is flat, and for* $R := S(V^*)$ *we have*

$$\operatorname{depth}\left(R^G\right) = k + \min\left\{2 + \sum_{i=1}^k \left[\frac{n_i}{2}\right], n_1 + \cdots + n_k\right\}.$$

Proof. From Lemma A.1 we obtain that $P_1(\mathrm{SO}_3(q), U)$ and $P_2(\mathrm{SO}_3(q), U)$ hold. Choosing the form as $Q(x_1, x_2, x_3) = x_1 x_3 - x_2^2$, we see that a Sylow p-subgroup of $\mathrm{SO}_3(q)$ is

$$P = \left\{ \begin{pmatrix} 1 & 2x & x^2 \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} \mid x \in \mathbb{F}_q \right\}$$

(see Taylor [56, p.142]), which is abelian. Moreover, the normalizer $\mathcal{N}_{\mathrm{SO}_3(q)}(P)$ is the semidirect product of P and the subgroup

$$H = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a^{-1} \end{pmatrix} \mid a \in \mathbb{F}_q^\times \right\},$$

which yields $P_4(\mathrm{SO}_3(q), U)$. Now from Proposition 4.3(c) we get the properties $P_1(G, U)$, $P_2(G, U)$, and $P_4(G, U)$. The first two properties are purely group-theoretic, and we have $P_4(G, V)$ by Proposition 4.3(d). To see that (G, V) is visibly flat, we only have to show that $P_3^+(\mathrm{SO}_3(q), V)$ holds and apply Proposition 4.3(c) again. Again we have to look at the Hilbert series $H\left(S(U)^\tau, t\right)$ for $1 \neq \tau \in P$, and compare this to $H\left(S(U)^P, t\right)$. Since τ has the Jordan canonical form $\left(\begin{smallmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{smallmatrix}\right)$, we know by Neusel [44] that the invariant ring $S(U)^\tau$ is generated by invariants of degrees $1, 2, p$, and p, the first three of which are algebraically independent. On the other hand, $S(U)^P$ has the invariants x_1 and Q (the quadratic form), which are algebraically independent. It follows that the dimensions of the invariants of $S(U)^P$ and $S(U)^\tau$ of degree $n < p$ coincide. More precisely,

$$\dim_K\left(S^n(U)^P\right) = 1 + \left[\frac{n}{2}\right]. \tag{4.5}$$

Thus $P_3^+(\mathrm{SO}_3(q), S^n(U))$ holds for $0 \leq n < p$, and $P_3^+(\mathrm{SO}_3(q), V)$ follows by Proposition 4.3(e).

We have shown that (G, V) is visibly flat, hence Theorem 4.2 yields

$$\operatorname{depth}\left(R^G\right) = \min\left\{\dim_K(V^P) + 2, \dim_K(V)\right\}.$$

Now the result follows from Equation (4.5). □

Remark 4.9. The number m in Theorem 4.8 is a divisor of $q - 1$. This follows as in Remark 4.5. ◁

We proceed by considering subgroups of $\mathrm{SU}_3(q)$, which is a further example of a finite group with a BN-pair of rank 1. By Theorem A.8 the only irreducible representation V such that $(\mathrm{SU}_3(q), V)$ is visibly flat is the natural one. However, we also obtain formulas for the depth of vector invariants.

Theorem 4.10. *Let* G *be a subgroup of* $\mathrm{SU}_3(q)$ *such that* p *divides* $|G|$, *where* $q = p^r$, *and let* U *be the natural three-dimensional* $K(\mathrm{SU}_3(q))$*-module, where* $\mathbb{F}_{q^2} \subseteq K$. *Let* k *be a positive integer and set* $V := \bigoplus_{i=1}^k U$ *and* $R := S(V^*)$. *Then* (G, V) *is flat, and*

$$\operatorname{depth}(R^G) = \min\left\{2 + k \cdot \dim_K(U^P), 3k\right\},$$

where $P \leq G$ *is a Sylow* p*-subgroup.*

Proof. By Theorem A.8, $(\mathrm{SU}_3(q), U)$ is visibly flat. Let $\widetilde{P} \leq \mathrm{SU}_3(q)$ be the Sylow p-subgroup of $\mathrm{SU}_3(q)$ given in Remark A.4(b). $\widetilde{P}$ is not abelian, and we do not have $P_3^+(\mathrm{SU}_3(q), U)$, so we have to modify our arguments somewhat to achieve the descent to G. $P_1(G, U)$ follows from $P_1(\mathrm{SU}_3(q), U)$, since we did not use the hypothesis that $\widetilde{P}$ is abelian in the proof of this statement in Proposition 4.3. Replacing G by a conjugate subgroup in $\mathrm{SU}_3(q)$, we can choose the Sylow p-subgroup of G to be $P = G \cap \widetilde{P}$, and from Equation (4.1) we obtain

$$H := \mathcal{N}_G(P) = G \cap \widetilde{H}.$$

Therefore H/P is embedded into $\widetilde{H}/\widetilde{P}$, hence it is abelian and acts faithfully on U^P. This gives $P_2(G, U)$ and $P_4(G, U)$. An element $\tau \in P$ with $\dim_K(U^\tau) > 1$ must be a matrix $\tau_{u,v}$ as in Remark A.4(b) with $u = 0$. The $\tau_{0,v} \in P$ form a subgroup $P_0 \leq P$. We have to consider two cases: either P_0 is a proper subgroup of P, or $P_0 = P$. In the first case, $\dim_K(U^P) = 1$, hence P_0 is the subgroup generated by all $\tau \in P$ with $U^\tau \supsetneqq U^P$, so we have $P_3(G, U)$. On the other hand, if $P_0 = P$, then $\dim_K(U^P) = 2$, and the only $\tau \in P$ with $U^\tau \supsetneqq U^P$ is $\tau_{0,0} = 1$, so we have $P_3^+(G, U)$. Hence $P_3(G, U)$ holds in any case. Now clearly all properties P_1–P_4 carry over to a direct sum of k copies of U, and we obtain that (G, V) is visibly flat. Thus the result follows from Theorem 4.2. □

4.4 Suzuki groups, Ree groups, and some other simple groups

Implicit in Theorems 4.4, 4.8, and 3.3 are results on the depths of invariant rings of the simple groups $\mathrm{PSL}_2(q)$ and A_n $(n < 2p)$ with various representations. We now look two other classes of simple groups, the Suzuki groups and the Ree groups.

Theorem 4.11. *Let G be a Suzuki group $\mathrm{Sz}(q)$ $(q = 2^{2m+1})$ or a Ree group $R(q)$ $(q = 3^{2m+1})$, and let U be the four or 7-dimensional natural KG-module, where K is of characteristic 2 or 3, respectively. Set $V := \bigoplus_{i=1}^k U$ with k a positive integer, and $R := S(V^*)$. Then (G, V) is flat, and*

$$\operatorname{depth}\left(R^G\right) = k + 2.$$

In particular, the invariant ring with respect to the natural module is not Cohen-Macaulay.

Proof. By Theorem A.8, (G, U) is visibly flat, and therefore also (G, V). The result now follows from Theorem 4.2 since $\dim(U^P) = 1$ (see Remark A.4). □

Campbell et al. [14, Example 6(b)] considered invariants of Sylow 2-subgroups of the Suzuki groups and found that the depth of the vector invariants of k copies of the natural representation is $k + 2$. However, they were able to determine the depth of the invariants of the Suzuki groups only for $\mathrm{Sz}(2)$ and $k = 1$.

Modular representations for a considerable number of simple groups can be downloaded from Robert Wilson's home page [57]. Using the computer algebra system MAGMA (see Bosma et al. [7]), I examined those sporadic groups which have a representation (available on [57]) of exceptional dimension over a field of characteristic p (meaning that a representation of this dimension does not exist for other p's), such that p^2 does not divide the group order. Attempts to look at other representations V showed that (G, V) tends not to be visibly flat. The instances of visibly flat pairs I found are:

- The Mathieu group M_{11} with the 9-dimensional representation over $\mathbb{F}_{11}$. The Sylow p-subgroup P has order p, so by Proposition 4.3(b) we only have to check property P_4, which is easily verified. Since the fixed space of P has dimension 1, the depth of the invariant ring is 3.

- M_{12} with the 29-dimensional representation over $\mathbb{F}_{11}$. Here the fixed space of the Sylow p-subgroup has dimension 3, so the depth is 5.

- M_{22} with the 20-dimensional representation over $\mathbb{F}_{11}$. The fixed space of the Sylow p-subgroup has dimension 2, so the depth is 4.
- M_{23} with the 21-dimensional representation over $\mathbb{F}_{23}$. The fixed space of the Sylow p-subgroup is one-dimensional, so the depth of the invariant ring is 3.
- The Janko group J_1 with the 7-dimensional representation over $\mathbb{F}_{11}$. Again the depth is 3.

5 More about GL_2 and SL_2

As we have seen in Section 4.2, the action of $\mathrm{SL}_2(q)$ on binary forms of degree n produces a visibly flat pair if $\gcd(n, q-1) \leq 2$. On the other hand, actions on binary forms of degrees not satisfying this condition, and furthermore almost all actions of the general linear group $\mathrm{GL}_2(q)$, are not visibly flat. However, this need not mean that we are unable to determine the depth of the invariant ring. For instance, in Example 4.6 we found that for the action of $\mathrm{SL}_2(p)$ on binary forms of degree $p-1$, the invariant ring is in fact Cohen-Macaulay and therefore has depth p.

5.1 Computing tables of depths

If we restrict our attention to the cases $G = \mathrm{SL}_2(p)$ or $G = \mathrm{GL}_2(p)$, we can use the methods discussed at the beginning of Section 3 to determine the depth of the invariant ring. Let U be the natural KG-module and $V := S^n(U)$ with $n < p$. V is an irreducible KG-module, and in the case $G = \mathrm{SL}_2(p)$ these are all irreducible modules (see Alperin [2, pp. 14–16]). On the other hand, $S^n(U)$ becomes decomposable as a KG-module if $n \geq p$, which we excluded. To avoid confusion and to follow the conventions, we will consider invariants in $R := S(V)$ rather than $S(V^*)$. To determine the depth of R^G by using Theorem 3.1, we have to find the smallest positive number i such that $H^i(H, R) \neq 0$, where $H := \mathcal{N}_G(P)$ is the normalizer of a Sylow p-subgroup $P \leq G$. But we know from (2.5) that $H^*(H, R) = H^*(P, R)^{H/P}$. Moreover, P is cyclic of order p, so $H^i(P, R)$ and the H/P-action are given by (3.1) and Proposition 3.2. However, the formulas (3.1) seem to suggest that (among other things) we have to compute the entire invariant ring R^P to obtain $H^i(P, R)$. This can in fact be avoided by using the following "periodicity" result, which can be found in Almkvist and Fossum [1, Proposition 2.4, p. 42] or Hughes and Kemper [29, Lemmas 2.9, 2.10]. Namely, for $d \geq p - n$ we have an isomorphism of KP-modules

$$S^d(V) \cong S^{d-p}(V) \oplus (\text{a free } KP\text{-module}),$$

where we formally set $S^{d-p}(V) := 0$ if $d - p < 0$. For $d \geq p - n$ and $i > 0$, this yields an isomorphism

$$H^i\left(P, S^d(V)\right) \cong H^i\left(P, S^{d-p}(V)\right) \tag{5.1}$$

of vector spaces. We can therefore restrict ourselves to the computation of $H^i\left(P, S^d(V)\right)$ for $0 \leq d < p - n$ by using (3.1), which only requires methods of linear algebra, and then use (5.1) to decide whether there exists *any* non-negative d for which $H^i\left(P, S^d(V)\right)^{H/P} \neq 0$. Since $H^i(H, R)$ is the direct sum of the $H^i\left(H, S^d(V)\right)$, this is equivalent to the condition that $H^i(H, R) \neq 0$. However, the isomorphism (5.1) does not preserve the H/P-action, so we need to give a more explicit description in order to understand the H/P-actions on both sides. Choose a basis x, y of U such that P is generated by the linear map sending y to $x + y$ and x to itself, and take $x_i := x^{n-i}y^i$ $(i = 0, \ldots, n)$ as a basis of V. Then with $N := \prod_{\sigma \in P} \sigma(x_n)$ we have

$$S(V) = N \cdot S(V) \oplus B$$

with B a KP-module (see Hughes and Kemper [29, Lemma 2.9]), and the degree-d part of B is free over KP if $d \geq p-n$ (see [29, Lemma 2.10]). From this the isomorphism (5.1) follows, but now we can see what the H/P-action on both sides is. Indeed, the matrix $\rho_a := \left(\begin{smallmatrix} a^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{GL}_2(p)$ acts trivially on x_n, and the matrix $\tau_a := \left(\begin{smallmatrix} 1 & 0 \\ 0 & a \end{smallmatrix}\right)$ acts by $\tau(x_n) = a^n x_n$. The ρ_a and τ_a together with P generate the normalizer of P in $\mathrm{GL}_2(p)$, hence we can define a linear character $\chi_n\colon \mathcal{N}_{\mathrm{GL}_2(p)}(P)/P \to \mathbb{F}_p^\times$ by $\chi_n(\rho_a P) = 1$, $\chi_n(\tau_a P) = a^n$. Then for $\tau \in H$ we have

$$\tau(N) = \prod_{\sigma \in P} \tau\sigma(x_n) = \prod_{\sigma \in P} \tau\sigma\tau^{-1}\left(\chi_n(\tau P) x_n\right) = \chi_n(\tau P)^p \cdot N = \chi_n(\tau P) \cdot N. \tag{5.2}$$

By abuse of notation, we write χ_n for the one-dimensional $K(H/P)$-module given by χ_n. Now for $d \geq p - n$ and $i > 0$ we obtain an isomorphism

$$H^i\left(P, S^d(V)\right) \cong \chi_n \otimes_K H^i\left(P, S^{d-p}(V)\right)$$

of $K(H/P)$-modules. It is therefore sufficient to form $H^i\left(P, S^d(V)\right)$ for $0 \leq d < p - n$ and to test whether in these $K(H/P)$-modules (with the action given by Proposition 3.2) there exists a non-zero vector on which H/P acts by a suitable power of the character χ_n. More explicitly, let $z \in \mathbb{F}_p$ be a generator of the multiplicative group $\mathbb{F}_p^\times$, and $\sigma = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Then we have the following criterion.

Proposition 5.1. *In the above situation, take $i \in \mathbb{N}_0$ and $\epsilon \in \{1, 2\}$. Then $H^{2i+\epsilon}(H, R)$ is non-zero if and only if for some $d \in \{0, \ldots, p - n - 1\}$ there exists an*

$$\alpha \in V_{\epsilon,d} := \begin{cases} S^d(V)^\sigma \big/ (\sigma - 1)^{p-1} S^d(V) & \text{if } \epsilon = 2, \\ \ker(\sigma - 1)^{p-1} \big/ (\sigma - 1) S^d(V) & \text{if } \epsilon = 1 \end{cases}$$

such that

- *for $G = \mathrm{SL}_2(p)$: α is an eigenvector of $\rho_z \tau_z$ with respect to an eigenvalue $z^{nj-2i-2}$ for some j,*
- *for $G = \mathrm{GL}_2(p)$: α is simultaneously an eigenvector of ρ_z with respect to the eigenvalue z^{-i-1} and an eigenvector of τ_z with respect to an eigenvalue z^{nj-i-1} for some j.*

The proof was given in the above discussion. From Proposition 5.1 it becomes apparent why we always get non-trivial first cohomology in the case $G = \mathrm{SL}_2(p)$ if $\gcd\{n, p - 1\} \leq 2$: then we can take $d = 0$ and $\alpha = 1$, which is an eigenvector for $1 = z^0$, and we have $0 = nj - 2$ for some j. This is exactly the technique used in the proof of Theorem 4.2. It is also clear from the above condition for $\mathrm{GL}_2(p)$ that this technique cannot work for $G = \mathrm{GL}_2(p)$. So to construct non-zero elements in the first cohomology in the situation of visible flatness, we always start with cohomology with values in $S^0(V) = K$ and move into values in R only by multiplication with "N-like" elements. Thus the scope of the above method is much wider, but the trade-off is that it is harder to prove theorems with that setup. (However, we will obtain a general result in Theorem 5.4.)

The above method to obtain the depth has important advantages over the method by computing the invariant ring explicitly using algorithms from Kemper [33], and then determining its depth by standard techniques. The first advantage is that it is much faster and therefore reaches further. Indeed, with the computation of invariant rings one usually starts running into memory and time problems for values of n larger than 3. On the other hand, with the above method we can handle cases of n well over 10. A second advantage is that we can use "rational" methods. More precisely, consider the polynomial ring $R_0 := \mathbb{Z}[x_0, \ldots, x_n]$. The **weight** of a monomial $t = x_0^{e_0} \cdots x_n^{e_n}$ is defined as

$$\mathrm{wt}(t) := \sum_{i=0}^{n} i e_i$$

(see Hilbert [27, p. 22]). A polynomial in R_0 is called **isobaric** of weight w if all its monomials have weight w. Now we see from the definition of the matrices ρ_a and τ_a that if $f \in R_0$ is a homogeneous, isobaric polynomial of degree d and weight w, then for the reduction $\bar{f} \in R$ of f modulo p we have

$$\rho_a(\bar{f}) = a^{w-nd} \bar{f} \quad \text{and} \quad \tau_a(\bar{f}) = a^w \bar{f}. \tag{5.3}$$

Therefore $\bar{f}$ meets the condition for non-zero first cohomology in Proposition 5.1 for $G = \mathrm{SL}_2(p)$ *for all* p if $w \equiv -1 \mod n$. Hence if we have found such an f we only have to determine those primes p for which $(\sigma - 1)^{p-1}(\bar{f}) \neq 0$ or $\bar{f} \in (\sigma - 1)R$. If $\sigma_0 = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z})$, then $(\sigma_0 - 1)^k(f) = 0$ for some k, which can be taken as an upper bound for the primes p with $(\sigma - 1)^{p-1}(\bar{f}) \neq 0$. Furthermore, if f satisfies the additional condition that $f \notin (\sigma_0 - 1)R_0$, then we can put up the augmented matrix corresponding to the inhomogeneous linear system whose solvability decides whether $f \in (\sigma_0 - 1)R_0$, and if p does not divide the largest elementary divisor of this matrix, then also $\bar{f} \notin (\sigma - 1)R$. Therefore if we can find a homogeneous, isobaric polynomial $f \in R_0$ of weight $w \equiv -1 \mod n$ such that $f \notin (\sigma_0 - 1)R_0$, then $H^1(H, R) \neq 0$ and thus $\operatorname{depth}(R^G) = 3$ (where $G = \mathrm{SL}_2(p)$) for all primes except for a finite set of possible exceptions, which can be determined. We will see in the proof of Theorem 5.4 that such an f always exists. Hence for a given n we can determine the depth of the invariant ring of $\mathrm{SL}_2(p)$ acting on binary forms of degree n for all primes p. Again, this technique does not carry over to the case $G = \mathrm{GL}_2(p)$.

The above methods to obtain the depth were implemented by the author in MAGMA. With this implementation the Tables 5.1 and 5.2 were produced.

	$p=3$	$p=5$	$p=7$	$p=11$	$p=13$	$p=17$	$p=19$	$p=23$	$p=29$
$n=1$	2	$\rightarrow$							
$n=2$	3	$\rightarrow$							
$n=3$	-	3	4	3	$\rightarrow$				
$n=4$	-	5	3	$\rightarrow$					
$n=5$	-	-	3	$\rightarrow$					
$n=6$	-	-	7	3	$\rightarrow$				
$n=7$	-	-	-	3	$\rightarrow$				
$n=8$	-	-	-	3	5	3	$\rightarrow$		
$n=9$	-	-	-	3	4	3	$\rightarrow$		
$n=10$	-	-	-	11	3	$\rightarrow$			
$n=11$	-	-	-	-	3	$\rightarrow$			
$n=12$	-	-	-	-	13	5	3	$\rightarrow$	
$n=13$	-	-	-	-	-	3	$\rightarrow$		
$n=14$	-	-	-	-	-	3	$\rightarrow$		
$n=15$	-	-	-	-	-	3	4	3	$\rightarrow$
$n=16$	-	-	-	-	-	17	3	$\rightarrow$	
$n=17$	-	-	-	-	-	-	3	$\rightarrow$	

Table 5.1: Depth of the invariant ring of $SL_2(p)$ acting on binary forms of degree n.

In Table 5.1 arrows ($\rightarrow$) indicate that from this position on all depths are equal to 3 (or 2 in the very first row). Notice that for $p < n$ the representation is decomposable and does not appear in the tables. The cases $n = 3$ and 4 were done by Shank and Wehlau [49]. Table 5.1 confirms their result that for these n all invariant rings have depth 3, except in the cases $(n,p) \in \{(3,7),(4,5)\}$, where the invariant rings are Cohen-Macaulay. Shank and Wehlau posed the questions whether for each n one would see only one exceptional prime where the depth is not 3, and whether for this prime the invariant ring would always be Cohen-Macaulay. Both questions can now be answered negatively: for $(n,p) = (8,13)$ we see an invariant ring of depth > 3 which is not Cohen-Macaulay, and for $n = 12$ we have the two exceptional primes $p = 13$ and $p = 17$. We have already given an explanation why for $G = \mathrm{SL}_2(p)$ the depth is generically 3, and will prove this (in a more general setting) in Theorem 5.4. Table 5.2 prompts the following conjecture.

Conjecture 5.2. *(a) The invariant ring of* $\mathrm{GL}_2(p)$ *acting on binary forms of degree 3 is Cohen-Macaulay.*

$p =$	3	5	7	11	13	17	19	23	29	31	37	41	43	47	53
$n = 1$	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
$n = 2$	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3
$n = 3$	-	4	4	4	4	4	4	4	4	4	4	4	4	4	4
$n = 4$	-	5	5	5	4	4	3	3	3	3	3	3	3	3	3
$n = 5$	-	-	6	4	4	3	3	3	3	3	3	3	3	3	3
$n = 6$	-	-	7	5	4	3	3	3	3	3	3	3	3	3	3
$n = 7$	-	-	-	8	4	3	3	3	3	3	3	3	3	3	3
$n = 8$	-	-	-	9	9	3	3	3	3	3	3	3	3	3	3
$n = 9$	-	-	-	10	10	4	3	3	3	3	3	3	3	3	3
$n = 10$	-	-	-	11	11	5	3	3	3	3	3	3	3	3	3
$n = 11$	-	-	-	-	12	8	3	3	3	3	3	3	3	3	3
$n = 12$	-	-	-	-	13	13	5	3	3	3	3	3	3	3	3
$n = 13$	-	-	-	-	-	14	12	3	3	3	3	3	3	3	3
$n = 14$	-	-	-	-	-	15	15	3	3	3	3	3	3	3	3
$n = 15$	-	-	-	-	-	16	16	4	3	3	3	3	3	3	3
$n = 16$	-	-	-	-	-	17	17	9	3	3	3	3	3	3	3
$n = 17$	-	-	-	-	-	-	18	18	3	3	3	3	3	3	3

Table 5.2: Depth of the invariant ring of $\mathrm{GL}_2(p)$ acting on binary forms of degree n.

(b) For every $n > 3$ there exists an integer m such that for $p \geq m$ the invariant ring of $\mathrm{GL}_2(p)$ *acting on binary forms of degree n has depth 3.*

We remark that with the same techniques we can also compute the depths of the invariants of $\mathrm{GL}_2(p)$ acting on the modules $V_{n,k} := \det^k \otimes_K S^n(U)$, where $\det^k$ denotes the one-dimensional module given by the action by the k-th power of the determinant. One simply has to adjust Proposition 5.1 to this situation. The $V_{n,k}$ cover all simple $\mathrm{GL}_2(p)$-modules. For example, $V_{n,0}^* = V_{n,-n}$, and it is easy to see that one gets from $V_{n,0}$ to $V_{n,-n}$ by first applying the automorphism of $\mathrm{GL}_2(p)$ given by $\sigma \mapsto (\sigma^{-1})^{\mathbf{T}}$. Hence the invariant rings $S\left(V_{n,0}\right)^{\mathrm{GL}_2(p)}$ and $S\left(V_{n,0}^*\right)^{\mathrm{GL}_2(p)}$ are isomorphic. The computation of the table of depths corresponding to Table 5.2 was done for the modules $V_{n,1}$. The result bears a few differences from Table 5.2. Although the depth for $V_{3,1}$ seems to be 4 for most primes p, it is 3 for $p = 11, 31$, and 41. For $n > 3$, the depth for large p again appears to be 3. In all cases that were calculated, the depth for $V_{n,1}$ is less than or equal to the depth for $V_{n,0}$.

5.2 The generic depth of invariants of binary forms

Our next goal is to prove that the depth of the invariant ring of $\mathrm{SL}_2(q)$ acting on binary forms of degree n is "generically" 3, and to do this in a more general setting involving subgroups and direct sums of symmetric powers of the natural representation. For this purpose we will use rational methods (which already helped us to construct Table 5.1) in a more systematic way. Let K be a field and take $U := K^2$ with basis x, y, and $V_{n+1} := S^n(U)$ with basis $x_i := x^{n-i}y^i$ $(i = 0, \ldots, n)$. Consider the derivation

$$D := x_0 \frac{\partial}{\partial x_1} + 2x_1 \frac{\partial}{\partial x_2} + \cdots + n x_{n-1} \frac{\partial}{\partial x_n}$$

on $R := S(V_{n+1})$ (see Hilbert [27, p. 27]). We write $\sigma_t := \left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right)$ for $t \in K$ and $G_a := \{\sigma_t \mid t \in K\}$ (which acts on R), and assume for the moment that $\mathrm{char}(K) = 0$. Then for $f \in R$ we have the equivalence

$$\sigma_1(f) = f \quad \Longleftrightarrow \quad f \in R^{G_a} \quad \Longleftrightarrow \quad Df = 0. \tag{5.4}$$

Indeed, the right hand equivalence can be found in Hilbert [27, p. 26], and the left hand equivalence follows since σ_1 generates G_a as an algebraic group. Also observe that D does not change the degree of a homogeneous polynomial, but decreases the weight of an isobaric polynomial (see on page 37) by one. Part (a) of the following lemma was brought to my attention by Jim Shank.

Lemma 5.3. *In the above setting, assume that* $\mathrm{char}(K) = 0$. *Then we have*

(a) *If* $0 \neq g \in R^{G_a}$ *is homogeneous of degree* d *and isobaric of weight* w, *then there exists an* $f \in R$ *which is homogeneous of degree* d *and isobaric of weight* $nd - w$ *such that*

$$D^{nd-2w} f = g \quad \text{and} \quad f \notin DR.$$

(b) *Suppose that for* $g, f \in R$ *there exists a non-negative integer* k *such that* $D^k f = 0$ *and* $g \in D^k R$. *Then*

$$gf \in DR.$$

Proof. (a) For the derivation

$$\Delta := n x_1 \frac{\partial}{\partial x_0} + (n-1) x_2 \frac{\partial}{\partial x_1} + \cdots + x_n \frac{\partial}{\partial x_{n-1}}$$

we have the rule

$$\left(D\Delta^i - \Delta^i D\right) h = i(nd' - 2w' - i + 1)\Delta^{i-1} h \tag{5.5}$$

for a homogeneous, isobaric $h \in R$ of degree d' and weight w' (Hilbert [27, p. 37]). Furthermore, by Hilbert [27, p. 44], the smallest non-negative m such that $\Delta^m g = 0$ is $m = nd - 2w + 1$. We set

$$f := \Delta^{nd-2w} g.$$

Thus $f \neq 0$ is homogeneous and isobaric of degree d and weight $nd - w$, and $\Delta f = 0$. We claim that for $0 \leq j \leq nd - 2w$ we have

$$D^j f = \frac{(nd - 2w)!}{(nd - 2w - j)!} j! \cdot \Delta^{nd-2w-j} g \tag{5.6}$$

This is true for $j = 0$. For $j \geq 0$, proceeding by induction and using (5.5) and $Dg = 0$, we obtain

$$\begin{aligned} D^{j+1} f &= \frac{(nd - 2w)!}{(nd - 2w - j)!} j! \cdot D\Delta^{nd-2w-j} g = \\ & \frac{(nd - 2w)!}{(nd - 2w - j)!} j! \cdot (nd - 2w - j)(nd - 2w - (nd - 2w - j) + 1)\Delta^{nd-2w-j-1} g, \end{aligned}$$

which yields (5.6) for $j + 1$. In particular, for $j = nd - 2w$ we obtain

$$D^{nd-2w} f = \left((nd - 2w)!\right)^2 g,$$

Thus replacing f by a non-zero scalar multiple yields the desired relation $D^{nd-2w} f = g$.

Now assume by way of contradiction that there exists $h \in R$ such that $f = Dh$. Then h can be assumed to be homogeneous of degree d and isobaric of weight $nd - w + 1$. Let k be the least non-negative number with $\Delta^k h = 0$. Since $h \neq 0$, k is positive. On the other hand, we have

$$\left(D\Delta^k - \Delta^k D\right) h = D0 - \Delta^k f = -\Delta^{k-1}\Delta f = 0,$$

but by (5.5)

$$\left(D\Delta^k - \Delta^k D\right) h = k\left(nd - 2(nd - w + 1) - k + 1\right)\Delta^{k-1} h.$$

Thus

$$0 = nd - 2(nd - w + 1) - k + 1 = 2w - nd - 1 - k,$$

from which the contradiction $k = -(nd - 2w + 1) = -m \leq 0$ follows.

(b) Let $g = D^k h$. We claim that for $0 \le i \le k$ we have

$$D^i h \cdot D^{k-i} f \in DR,$$

which for $i = k$ gives part (b) of the lemma. For $i = 0$, we have $h \cdot D^k f = 0 \in DR$, so the claim is true. For $i > 0$, we have

$$D\left(D^{i-1}h \cdot D^{k-i} f\right) = D^i h \cdot D^{k-i} f + D^{i-1} h \cdot D^{k-(i-1)} f.$$

Solving this equation for $D^i h \cdot D^{k-i} f$ yields the claim by induction. □

We can now prove the theorem about the depth of invariants of subgroups of $\mathrm{SL}_2(q)$ for large characteristics. Notice that the formulas in the following theorem and in Theorem 4.4 are identical. The point is that we do no longer need the hypothesis made in Theorem 4.4 on the greatest common divisor, and instead exclude a finite number of primes.

Theorem 5.4. *Let $n_1, \ldots, n_k$ be non-negative integers. Then there exists an integer m such that for a finite field K of characteristic $p \ge m$ the following holds: If $G \le \mathrm{SL}_2(K)$ is a subgroup whose order is divisible by p, and $V = S^{n_1}(U) + \cdots + S^{n_k}(U)$ with U the natural $\mathrm{SL}_2(K)$-module, then (G, V) is flat (as defined on page 3), and*

$$\operatorname{depth}\left(S(V)^G\right) = k + \min\{2, n_1 + \cdots + n_k\}.$$

Proof. We may assume that $n_1 \ge 3$, since if all n_i are ≤ 2, the result follows from Theorem 4.4. If $p > n_i$ for all i, then

$$\dim_K\left((V^*)^P\right) = k$$

for a Sylow p-subgroup $P \le G$. With $H := \mathcal{N}_G(P)$ we have that H is strongly p-embedded and $H_{V^*\text{-sing}} = \{1\}$. Therefore the result follows from Corollary 2.9 if we can show that $H^1(H, S(V)) \ne 0$ for large p. For this it suffices to show that $H^1(H, R) \ne 0$, where $R := S\left(S^{n_1}(U)\right)$. In the sequel we write n for n_1.

We take the polynomial ring $R_0 := \mathbb{C}[x_0, \ldots, x_n]$ and use the notation of Lemma 5.3. By Hilbert [27, p. 64] there exists a homogeneous, isobaric polynomial $f_1 \in \mathbb{Q}[x_0, \ldots, x_n] \subset R_0$, of degree 3 and weight $n-1$ such that $Df_1 = 0$. Then $f_2 := (x_1^2 - x_0 x_2) f_1$ has degree 5 and weight $n+1$, and $Df_2 = 0$. By Lemma 5.3(a), there exists $f \in R_0$ of degree 5 and weight $4n-1$ such that

$$f \notin DR_0 \quad \text{and} \quad D^{3n-1} f = 0. \tag{5.7}$$

Take $g := x_0^3$ and $\hat{g} := (x_1^2 - x_0 x_2)^4$. Then again by Lemma 5.3(a) we have $g \in D^{3n} R_0$ and $\hat{g} \in D^{8n-16} R_0$. Since $8n - 16 \ge 3n - 1$, it follows from Lemma 5.3(b) that there exist homogeneous and isobaric $h, \hat{h} \in R_0$ such that $gf = Dh$ and $\hat{g} f = D\hat{h}$. Observe that all polynomials have coefficients in $\mathbb{Q}$, so multiplying f, h, and $\hat{h}$ by a non-zero integer, we can achieve that $h \in \mathbb{Z}[x_0, \ldots, x_n]$. We obtain

$$D(g\hat{h} - \hat{g}h) = g \cdot D\hat{h} - \hat{g} \cdot Dh = g\hat{g}f - \hat{g}gf = 0.$$

Hence by (5.4), $g\hat{h} - \hat{g}h$ is an invariant of the additive group $G_a \le \mathrm{SL}_2(\mathbb{C})$, so for $\sigma_t := \left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right) \in G_a$ we have

$$0 = (\sigma_t - 1)(g\hat{h} - \hat{g}h) = g \cdot (\sigma_t - 1)\hat{h} - \hat{g} \cdot (\sigma_t - 1)h.$$

By the coprimality of g and $\hat{g}$ this implies that g divides $(\sigma_t - 1)h$. We claim that

$$(\sigma_1 - 1)\frac{h}{g} \notin (\sigma_1 - 1)R_0.$$

Indeed, by way of contradiction, assume that $(\sigma_1 - 1)h = g \cdot (\sigma_1 - 1)\widetilde{h}$ with $\widetilde{h} \in R_0$. Then $\sigma_1(h - g\widetilde{h}) = h - g\widetilde{h}$, hence $D(h - g\widetilde{h}) = 0$ by (5.4). But this implies $f = D\widetilde{h}$, in contradiction to (5.7).

Whether or not $(\sigma_1 - 1)h \in g \cdot (\sigma_1 - 1)R_0$ depends on the rank of a certain matrix over $\mathbb{Z}$, which is obtained from equating $(\sigma_1 - 1)h$ to a linear combination (with unknown coefficients) of $g \cdot (\sigma_1 - 1)t$, where t ranges through the monomials of degree 5. If p does not divide the elementary divisors of this matrix, then

$$(\sigma_1 - 1)h \notin g \cdot (\sigma_1 - 1)\mathbb{Z}[x_0, \ldots, x_n] + p\mathbb{Z}[x_0, \ldots, x_n] \tag{5.8}$$

follows. Hence there exists a number m such that (5.8) holds for $p \geq m$. Now assume $\operatorname{char}(K) = p \geq m$ and choose a number field $K_0 \subset \mathbb{C}$ (e.g., a suitable cyclotomic field) such that for the integral closure $\mathcal{O} \subset K_0$ of $\mathbb{Z}$ in K_0 there exists a ring-epimorphism $\mathcal{O} \to K$. We denote the (componentwise) image of a matrix $\sigma \in \mathrm{SL}_2(\mathcal{O})$ in $\mathrm{SL}_2(K)$ by $\overline{\sigma}$. With a suitable choice of a basis of U we can assume that $\overline{\sigma_1} \in P$. The image of a polynomial $u \in \mathcal{O}[x_0, \ldots, x_n]$ in $K[x_0, \ldots, x_n] = R$ will also be written as $\overline{u}$. Since every $\sigma \in P$ has a preimage in $\mathrm{SL}_2(\mathcal{O})$, $\overline{g}$ divides $(\sigma - 1)\overline{h}$ for every $\sigma \in P$. Therefore the map

$$P \to R, \ \sigma \mapsto (\sigma - 1)\frac{\overline{h}}{\overline{g}}$$

defines an element $\alpha \in H^1(P, R)$. From (5.8) we conclude $\alpha \neq 0$. Now we consider the action of H/P on $H^1(P, R)$. For $\tau_a = \left(\begin{smallmatrix} a^{-1} & 0 \\ 0 & a \end{smallmatrix}\right) \in H$, $\tau_a(\alpha)$ is given by the map

$$\begin{aligned} P \to R, \ \sigma \mapsto \tau_a\alpha(\tau_a^{-1}\sigma\tau_a) &= \tau_a\left(\frac{(\tau_a^{-1}\sigma\tau_a - 1)\overline{h}}{\overline{g}}\right) = (\sigma - 1)\frac{\tau_a(\overline{h})}{\tau_a(\overline{g})} \\ &= a^{2\operatorname{wt}(h) - n\deg(h) + n\deg(g) - 2\operatorname{wt}(g)} \cdot \alpha(\sigma) = a^{3n}\alpha(\sigma), \end{aligned}$$

where we used (5.3). For $N := \prod_{\sigma \in P} \sigma(\overline{x_n})$ we get as in (5.2) that $\tau_a(N) = a^{n|P|} \cdot N$. Since $|P|$ and $[H : P]$ are coprime, there exists an integer e such that $a^{e|P|} = a^{-3}$, hence

$$N^e\alpha \in H^1(P, R)^{H/P} = H^1(H, R).$$

With $W := Kx_0 + \cdots + Kx_{n-1}$, P acts trivially on $S^n(U)/W$, and $N \notin (W)R$. Hence Proposition 1.2 applies and yields that multiplication by N is injective on $H^1(P, R)$. Therefore $N^e\alpha \neq 0$, which completes the proof. □

A Appendix: Visible flatness for irreducible modules of groups with BN-pair

by Kay Magaard

Throughout let G be a perfect group with a rank one BN-pair, defined over F_q, and let V be an absolutely irreducible G-module defined over the algebraic closure of F_q, $q = p^r$. Let P be a Sylow-p-subgroup of G. Recall that the pair (G, V) is said to be visibly flat if the properties (P_1)–(P_4) given in Definition 4.1 on page 29 are satisfied. Recall also that a sugroup X of a group H is T.I., if for all $h \in H$ either $X \cap X^h = 1$ or $X \cap X^h = X$.

Lemma A.1. *(a) The Sylow-p-subgroups of G are T.I., which implies that (P_1) holds.*

(b) $N_G(P)/C_G(P).P$ is cyclic, which implies that (P_2) holds.

Proof. By Curtis [17, Corollary 3.5] the Sylow p-subgroup P can be taken to be the group U from the definition of split BN-pairs. Moreover, the group B is $B = N_G(P)$ (see the proof of Proposition 2.5.1 in Carter [16]). Since the rank is one, the Weyl group W is of order 2. If $n_0 \in N$ is a representative of the non-trivial element in W, then

$$G = B \cup Bn_0B$$

by Carter [16, Propositions 2.1.1 and 2.1.2]. Therefore any $\sigma \in G \setminus N_G(P)$ can be written as $\sigma = bn_0b'$ with $b, b' \in B = N_G(P)$, and we obtain

$$^{b^{-1}}(P \cap {^\sigma P}) = P \cap {^{n_0}P} = \{1\}$$

by Carter [16, Proposition 2.5.5(i)]. From this (P_1) follows.

Part(b) is clear from the matrix descriptions of $H = N_G(P) = P \rtimes T$ given below. □

Remark. No groups with BN-pair rank two or more contain non-trivial strongly p-embedded subgroups, see for example Gorenstein et al. [26] for the case $p = 2$. In particular the Sylow-p subgroups of such a group is never T.I.. The latter is easy to see. Consider a parabolic subgroup whose Levi factor is a central product of an $SL_2(q)$ and a maximal split torus. Clearly two distinct Sylow-p subgroups of such a parabolic intersect in the radical of that parabolic. For the general case one can use the fact that any overgroup of P is contained in a parabolic subgroup of G. Then we can use the formula for the intersection of two conjugates of a parabolic subgroup given in Curtis and Reiner [18] to construct an intersection with the property that its order is divisible by p. ◁

Lemma A.2. *If P has exponent e and V is irreducible with $\dim(V) \geq e + 1$, then (P_3) does not hold.*

Proof. The elements of P are unipotent elements. Thus by picking a suitable basis, any element $g \in P$ can be brought into Jordan canonical from. Now if $\dim(V) \geq e + 1$ and $|g| \leq e$ then $\dim(V^g) \geq 2 > 1 = \dim(V^P)$ (Curtis [17, Theorem 4.3(c)]), and the claim follows. □

Lemma A.3. *The exponent e of P is as follows:*

G	$SL_2(q)$	$U_3(q), q - odd$	$R(q)$	$Sz(q)$	$U_3(q), q - even$
e	p	p	9	4	4

Proof. The groups in the table have faithful matrix representations over the algebraic closure of F_q of dimensions 2, 3, 7, 4, 3, respectively and hence have exponents at most as claimed. If the claimed exponent is a prime this completes the argument. For the other cases this follows by inspection of the matrix representations given below. In every case the group P contains a matrix whose Jordan canonical form is exactly one block of dimension equal to 7, 4 and 3, respectively. □

Remark A.4. In the following we give some information about the "natural" representation of G. For the different types of groups G, we list the matrices belonging to a general element from P and from a maximal torus T, all with an appropriate choice of a basis.

(a) $G = \mathrm{SL}_2(q)$:

$$\begin{aligned} P &= \{\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \mid s \in F_q\}, \\ T &= \{\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}, \mid t \in F_q^\star\}. \end{aligned}$$

(b) $G = \mathrm{SU}_3(q)$:

$$\begin{aligned} P &= \{\begin{pmatrix} 1 & u & v \\ 0 & 1 & -u^q \\ 0 & 0 & 1 \end{pmatrix} \mid u, v \in \mathbb{F}_{q^2} \text{ such that } u^{q+1} + v + v^q = 0\}, \\ T &= \{\begin{pmatrix} t & 0 & 0 \\ 0 & t^{q-1} & 0 \\ 0 & 0 & t^{-q} \end{pmatrix} \mid t \in F_{q^2}^\star\} \end{aligned}$$

(see Taylor [56, p. 121]).

(c) $G = \mathrm{Sz}(q)$, $q = 2^{2m+1}$:

$$\begin{aligned} P &= \{\begin{pmatrix} 1 & u & v & u^2u^\vartheta + uv + v^\vartheta \\ 0 & 1 & u^\vartheta & u^{\vartheta+1} + v \\ 0 & 0 & 1 & u \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid u, v \in F_q\}, \text{ where } \vartheta = 2^{m+1}, \\ T &= \{\begin{pmatrix} t^{1+2^m} & 0 & 0 & 0 \\ 0 & t^{2^m} & 0 & 0 \\ 0 & 0 & t^{-2^m} & 0 \\ 0 & 0 & 0 & t^{-1-2^m} \end{pmatrix} \mid t \in F_q^\star\} \end{aligned}$$

(see Huppert and Blackburn [32, XI, Lemma 3.1 and Theorem 3.3]).

(d) $G := R(q)$, $q = 3^{2m+1}$:

$$P = \{\begin{pmatrix} 1 & t^\vartheta & -u^\vartheta & (tu)^\vartheta - v^\vartheta & f_1(t,u,v) & f_2(t,u,v) & f_3(t,u,v) \\ 0 & 1 & t & u^\vartheta + t^{\vartheta+1} & -t^{2\vartheta+1} - v^\vartheta & f_4(t,u,v) & f_5(t,u,v) \\ 0 & 0 & 1 & t^\vartheta & -t^{2\vartheta} & v^\vartheta + (tu)^\vartheta & f_6(t,u,v) \\ 0 & 0 & 0 & 1 & t^\vartheta & u^\vartheta & (tu)^\vartheta - v^\vartheta \\ 0 & 0 & 0 & 0 & 1 & -t & u^\vartheta + t^{\vartheta+1} \\ 0 & 0 & 0 & 0 & 0 & 1 & -t^\vartheta \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \mid s, u, v \in F_q\},$$

where

$$\begin{aligned} \vartheta &= 3^m, \\ f_1(t,u,v) &= -u - t^{3\vartheta+1} - (tv)^\vartheta, \\ f_2(t,u,v) &= -v - (uv)^\vartheta - t^{3\vartheta+2} - t^\vartheta u^{2\vartheta}, \\ f_3(t,u,v) &= t^\vartheta v - u^{\vartheta+1} + t^{4\vartheta+2} - v^{2\vartheta} - t^{3\vartheta+1}u^\vartheta - (tuv)^\vartheta, \\ f_4(t,u,v) &= -u^{2\vartheta} + t^{\vartheta+1}u^\vartheta + tv^\vartheta, \\ f_5(t,u,v) &= v + tu - t^{2\vartheta+1}u^\vartheta - (uv)^\vartheta - t^{3\vartheta+2} - t^{\vartheta+1}v^\vartheta, \\ f_6(t,u,v) &= u + t^{3\vartheta+1} - (tv)^\vartheta - t^{2\vartheta}u^\vartheta. \end{aligned}$$

$$T = \{ \begin{pmatrix} t^{\vartheta} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & t^{1-\vartheta} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & t^{2\vartheta-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & t^{1-2\vartheta} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & t^{\vartheta-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & t^{-\vartheta} \end{pmatrix} \mid t \in F_q^{\star} \}$$

(see Kemper et al. [40]). ◁

From the Tensor Product Theorem of Steinberg [53] we recall that an irreducible G-module is either a p-restricted high weight module, or a twisted tensor product of such. A galois twist of a representation ρ is a representation $\alpha \circ \rho$ where α is a field automorphism of F_q. The number of (galois-) twists of a fixed module is equal to $\log_p(q)$. Now a weight is a map from a fixed maximal split torus of G into the algebraic closure of F_q. In our case such a maximal split torus is simply a complement to a Sylow-p-subgroup inside its normalizer, i.e., T as above. Moreover any such torus is a cyclic group.

Lemma A.5. (P_3) *does not hold for irreducible proper [all factors nontrivial] twisted tensor products.*

Proof. Suppose that V and W are irreducible G modules. Without loss of generality we may assume that (P_3) holds for both V and W, otherwise (P_3) clearly fails. Now by Magaard and Gluck [42, Proposition 1.2.1.1], we get that $\dim(V \otimes W)^g \geq \min(\dim(V), \dim(W)) \geq 2$ for all $g \in P$. On the other hand if $V \otimes W$ is a twisted tensor product, hence irreducible, we must have $\dim(V \otimes W)^P = 1$. The claim follows. □

Lemma A.6. *If V is an (absolutely)-irreducible G-module with $\dim(V) \geq 2$ for which (P_3) holds, then (G, V) are as follows:*

(a) $G = SL_2(q)$ and $V = Sym^a(N)$ where $a \leq p-1$ and N is a galois twist of the natural module of G.

(b) $G = U_3(q)$, and V is a galois-twist of the natural module or its dual.

(c) $G = Sz(q)$, and V is one of the 4-dimensional modules.

(d) $G = R(q)$, and V is one of the 7-dimensional modules.

Moreover all modules on the list have the (P_3) property.

Proof. The p-restricted high weight modules of $SL_2(q)$ are as in the conclusion. They all have the property that a nonidentity element of P acts via a single Jordan block, see Carter [15, Lemma 6.1.2]. If G is a Suzuki group, then the exponent of P is 4 and hence only modules of dimension 4 or less can be (P_3). Every 4-dimensional module of a Suzuki group is a twist of the natural module. The matrices of P acting on the natural module are given in Remark A.4. The (P_3) property can be gleaned from there.

The exponent of P for a Ree group is 9 and thus only the modules of dimension 9 or less can possibly have the (P_3) property. It is well known that any irreducible module of dimension ≥ 8 must have dimension at least 14 (see Gilkey and Seitz [25]). Thus only a 7-dimensional module can possibly have the (P_3) property. Now all 7-dimensional modules of a Ree group are twists of the "natural" 7-dimensional module. Now the matrix representation of P is given in Remark A.4. From it we infer that the elements of order 3 in P form the subgroup of P all of whose elements have $s = 0$. We also see that any element g outside that subgroup has the property that the rank of $g - 1$ is six. This shows that (P_3) holds.

If G is unitary we first establish a lower bound for an irreducible high weight module $V(\lambda)$ of weight $\lambda = m_1\lambda_1 + m_2\lambda_2$, where the λ_i are the fundamental weights. By the general representation theory (Gilkey and Seitz [25], Burgoyne and Williamson [10]) we have that $\dim(V(\lambda)) = \sum_{\mu<\lambda}[W : W(\mu)]m_\mu$ where μ ranges over the weights subdominant to λ (that is $\lambda - \mu$ is a positive integral combination of positive roots and μ is a positive integral combination of the λ_i), m_μ is the multiplicity of the weight μ, W is the Weyl group of G ($= Sym_3$ in this case), and $W(\mu)$ is the stabilizer of μ in W. Now a result of Suprunenko [55], as quoted and extended by Premet [47] shows that $m_\mu \neq 0$ for any subdominant weight μ. We now bound the number S of subdominant weights of λ with the property that $W(\mu) = 1$ from below. We note that $W(\mu) = 1$ iff $\mu = a_1\lambda_1 + a_2\lambda_2$ with $a_1 \neq 0 \neq a_2$. By Humphreys [30, page 68], for example, we see that $\alpha_1 = 2\lambda_1 - \lambda_2$ and that $\alpha_2 = -\lambda_1 + 2\lambda_2$, where the α_i are the fundamental roots of the A_2-type root system. Consequently $\alpha_1 + \alpha_2 = \lambda_1 + \lambda_2$. Thus we see that for the purposes of our argument the roles of m_1 and m_2 are symmetric. Therefore we may assume that $m_1 \geq m_2$. Now we compute that $\lambda - b\alpha_1 = (m_1 - 2b)\lambda_1 + (m_2 + b)\lambda_2$. So for every $1 \leq b \leq [\frac{m_1-m_2}{3}]$ we get a subdominant weight μ with $W(\mu) = 1$. Now we observe that for every $1 \leq c \leq (m_2 + b)$ we get a subdominant weight distinct from the previous set with the property that it has trivial Weyl group stabilizer; namely $\lambda - c\alpha_1 - c\alpha_2$. Thus we get that $S \geq [\frac{m_1-m_2}{3}] + (m_2 + [\frac{m_1-m_2}{3}]) \geq m_2 + 2[\frac{m_1-m_2}{3}]$. So $\dim(V(\lambda)) \geq 6S \geq 6m_2 + 12(m_1 - m_2 - 3)/3 \geq 4m_1 + 2m_2 - 12$.

Now $V(\lambda)$ is a section of the $(m_1 + m_2)$-fold tensor product of copies of the natural module and its dual. For characteristic zero this is exercise 8, page 117 of Humphreys [30]. The finite characteristic case follows by reduction mod p. Using the filtration of the tensor product in the proof of proposition 1.2.1 from Magaard and Gluck [42], we see that a Jordan block of a p-element acting on $V \otimes W$ is at most $\dim(V) + \dim(W) - 1$. It follows by induction on $m_1 + m_2$ that the size of a Jordan block of a p-element of G acting on $V(\lambda)$ is at most $2(m_1 + m_2) + 1$.

If $m_1 \geq 7$, then $4m_1 + 2m_2 - 12 > 2m_1 + 2m_2 + 1$. So then $\dim(V^g) \geq 2$ for all $g \in P$ showing that (P_3) can not hold.

Now we establish sharper lower bounds for $\dim(V(\lambda))$ when $m_1 \leq 6$.

We compute that $\lambda - a_1\alpha_1 - a_2\alpha_2 = (m_1 - 2a_1 + a_2)\lambda_1 + (m_2 + a_1 - 2a_2)\lambda_2$.

Now if $m_2 \geq 4$, then the weights $\lambda - a\alpha_1 - a\alpha_2$ for $1 \leq a \leq 3$ are subdominant and also the weights $\lambda - a\alpha_2$ where $1 \leq a \leq 2$. Now all but possibly the weight $\lambda - 2\alpha_2$ have trivial Weyl group stabilizer, whereas the order for the Weyl group stabilizer of $\lambda - 2\alpha_2$ is at most two. Thus in this case $\dim(V(\lambda)) \geq 6.4 + 3 = 27$. However under our assumptions, the size of a Jordan block is never more than 25; again establishing that (P_3) does not hold. So now we may also assume that $m_2 \leq 3$, reducing the upper bound for the size of a Jordan block to 19. If $m_1 \geq 4$ and $m_2 \neq 0$, then the weights λ, $\lambda - \alpha_1$, $\lambda - 2\alpha_1$, and $\lambda - \alpha_1 - \alpha_2$ are subdominant, guaranteeing that $\dim(V(\lambda)) \geq 18 + 3 = 21 > 19$, again showing that (P_3) can not hold. Similarly if $m_1 \geq 5$ and $m_2 = 0$, then the weights λ, $\lambda - \alpha_1$, $\lambda - 2\alpha_1$, and $\lambda - 2\alpha_1 - \alpha_2$ are subdominant, guaranteeing that $\dim(V(\lambda)) \geq 18 + 3 = 21 > 19$, again showing that (P_3) can not hold. So now we may assume that either $\lambda = 4\lambda_1$ or that $m_1 \leq 3$. In the first case $\dim(V(\lambda)) = 15$ whereas the upper bound for the size of a Jordan block is 9, again showing that (P_3) does not hold. So now we assume that $m_1 \leq 3$. If $m_2 = 3$, then $\dim(V(\lambda)) \geq 18 > 13$, again showing that (P_3) does not hold. So without loss of generality $m_2 \leq 2$, showing that the size of a Jordan block is at most 11. Now if $(m_1, m_2) = (3, 2), (3, 1), (2, 2), (2, 1)$, then $\dim(V(\lambda)) \geq 12 > 11$, again showing that (P_3) does not hold. Now $\dim(V(3\lambda_1)) = 10$ whereas in this case the upper bound for a Jordan block is 7. Now if $(m_1, m_2) = (2, 0), (1, 1)$, then $\dim(V(\lambda)) \geq 6$, whereas the upper bound for the size of a Jordan block is 5, again showing that (P_3) does not hold. This leaves $\lambda = \lambda_1$ as the only possibility; i.e that $V(\lambda)$ is the natural G-module. It is clear, see Remark A.4(b), that the natural G-module satisfies (P_3). To get the conclusion of the lemma we have to allow for duality and galois-twists (see Steinberg's Tensor Product Theorem). □

Lemma A.7. *If (G, V) occurs in conclusion (b),(c) or (d) of the previous lemma, then (P_4) holds. If (G, V) occurs in conclusion (a) of the previous lemma, then (P_4) holds iff $(a, q - 1) \leq 2$.*

Proof. The first part follows by inspection. For the second part we note that action of H on $C_V(P)$ is given by the map $h \mapsto h^a$ and that the center of G acts trivially on V iff $a = 2$. □

Remark. If the subgroup T is not cyclic, then property (P_4) can not possibly hold. In particular no group of Lie type of BN-pair rank two or more can have a representation satisfying (P_4). ◁

Combining the previous two lemmas with the first lemma and the remark after the first lemma yields the following.

Theorem A.8. *If G is a perfect group with BN-pair and V is an irreducible G-module in defining characteristic, then (G, V) is visibly flat iff*

(a) $G = SL_2(q)$ and $V = Sym^a(N)$, where $(a, q-1) \leq 2$ and N is a galois-twist of the natural module.

(b) $G = U_3(q)$ and V is a galois-twist of the natural module or its dual.

(c) $G = Sz(q)$ and V is a galois twist of the natural 4-dimensional module.

(d) $G = R(q)$ and V is a galois twist of the natural 7-dimensional module.

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Gregor Kemper
Department of Mathematics and Statistics
Queen's University
Kingston, Ontario K7L 3N6
Canada
`Gregor.Kemper@iwr.uni-heidelberg.de`

Kay Magaard
Department of Mathematics
Wayne State University
Detroit, Michigan 48202
USA
`kaym@math.wayne.edu`