Properties of Conditional Expectation

Definition: Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space, X a random variable with $E[X] < \infty$ and $\mathcal{G} \subset \mathcal{F}$ a sub-σ-algebra. Then $Y = E[X|\mathcal{G}]$ is the conditional expectation of X w.r.t $\mathcal{G}$ if:

1. Y is measurable w.r.t. $\mathcal{G}$.
2. For all bounded, $\mathcal{G}$-measurable random variables Z: $E[YZ] = E[XZ]$.

Notation: Let X be a RV with $E[X] < \infty$ and Y a RV. Then $E[X|Y] := E[X|\sigma(Y)]$.

Properties: We denote by $(\Omega, \mathcal{F}, \mathcal{P})$ a probability space, X, Y random variables with expectations and $\mathcal{A} \subset \mathcal{G} \subset \mathcal{F}$ sub-σ-algebras.

i) **Existence:** The conditional expectation $E[X|\mathcal{G}]$ exists and is unique up to measure zero.

ii) **Measurable Random Variables:** If X is measurable w.r.t. $\mathcal{G}$, then: $E[X|\mathcal{G}] = X$. If in addition $E[|XY|] < \infty$, then: $E[XY|\mathcal{G}] = XE[Y|\mathcal{G}]$.

iii) **Independence:** If $\sigma(X)$ and $\mathcal{G}$ are independent, then: $E[X|\mathcal{G}] = E[X]$. I.e. $E[XY|\mathcal{G}] = E[X]E[Y|\mathcal{G}]$ if X, Y are independent.

iv) **Trivial σ-algebra:** $E[X|\{\emptyset, \Omega\}] = E[X]$.

v) **Tower Property:** $E[E[X|\mathcal{A}]\mathcal{G}] = E[E[X|\mathcal{G}]\mathcal{A}] = E[X|\mathcal{A}]$. And as an immediate consequence of iv) and v): $E[E[X|\mathcal{G}]] = E[X]$.

vi) **Linearity:** $E[\lambda X + Y|\mathcal{G}] = \lambda E[X|\mathcal{G}] + E[Y|\mathcal{G}]$ for $\lambda \in \mathbb{R}$.

vii) **Monotonicity:** If $X \geq Y$ a.s., then $E[X|\mathcal{G}] \geq E[Y|\mathcal{G}]$.

viii) **Jensens Inequality:** For $f : \mathbb{R} \to \mathbb{R}$ convex: $E[f(X)|\mathcal{G}] \geq f(E[X|\mathcal{G}])$, if the expectations exists (i.e. $E[f(X)] < \infty$).

Note: Dominated convergence, monotone convergence and Fatous Lemma can be used in similar manner with conditional expectation.

Heuristic Remark: In contrast to the expectation (which is constant) the conditional expectation is a random variable. The conditional expectation is "the best guess" given the knowledge of a σ-algebra $\mathcal{G}$. By "knowing" a σ-algebra it is meant that for every $A \in \mathcal{G}$ you know if A happens or not. Thus if a random variable X is measurable w.r.t to $\mathcal{G}$ and you know $\mathcal{G}$ then you know X, too. If X is independent of $\mathcal{G}$ then the σ-algebra $\mathcal{G}$ gives no additional information about X. The trivial σ-algebra provides no information, too.

This sheet is only meant to give an overview of properties of the conditional expectation and is only a personal helpsheet. Neither completeness nor correctness can be guaranteed.