Additional Information:
For the whole exam, let $(\Omega, \mathcal{F}, \mathsf{P})$ be a probability space and denote by E the corresponding expectation. A random variable X is Poisson distributed with parameter $\lambda > 0$ if $\Omega = \{0, 1, 2, \ldots\}$ and $\mathsf{P}(X = k) = \lambda^k e^{-\lambda}/k!$. A random variable is standard normally distributed if $\Omega = \mathbb{R}$ and it has density $x \to e^{-x^2/2}/\sqrt{2\pi}$.

Exercises

Problem 1

Let $X_1, X_2, \ldots$ be independent and identically distributed random variables on the same probability space such that $\mathsf{P}(|X_1| < \infty) = 1$.

(i) Show that $\frac{1}{n} X_n$ converges to 0 in probability.

(ii) Now assume that X_k has density $f(x) = Cx^{-2}\,\mathbb{1}_{\{|x|>1\}}$ for all $k \in \mathbb{N}$, where $\mathbb{1}_A$ denotes the indicator function of a set A and C is a normalizing constant such that f is a density. Show that $\frac{1}{n} X_n$ does not converge almost surely.

Problem 2

Let $X, Y_1, Y_2, \ldots$ be independent random variables, where X is Poisson distributed with parameter 1 and Y_k is standard normally distributed for all $k \geq 1$. Calculate the characteristic function of $S := \sum_{k=1}^{X} Y_k$.

Problem 3

Let $\Omega = [-1/2, 1/2]$, $\mathcal{F} = \mathcal{B}([-1/2, 1/2])$ be the Borel-σ-algebra on Ω and P be the Lebesgue measure. Define two random variables $X, Y \mapsto \mathcal{B}(\mathbb{R})$ by

$$X(\omega) = \omega^2 \quad \text{and} \quad Y(\omega) = \omega^3$$

for all $\omega \in \Omega$. Calculate the following conditional expectations,

(i) $\mathsf{E}[X|Y] := \mathsf{E}[X|\sigma(Y)]$ and

(ii) $\mathsf{E}[Y|X] := \mathsf{E}[Y|\sigma(X)]$,

where $\sigma(X)$ and $\sigma(Y)$ denote the σ-algebras generated by X and Y respectively.
Remark: For full points you need to provide either a calculation or an argument.

Please turn the page!

Problem 4

Let $X_1, X_2, \ldots$ be independent and identically distributed random variables on the same probability space such that

$$\mathsf{P}\left(X_1 = -\frac{1}{4}\right) = \mathsf{P}\left(X_1 = \frac{1}{4}\right) = \frac{1}{2}.$$

For all $n \in \mathbb{N}$ define

$$M_n := \prod_{k=1}^{n} (1 + X_k).$$

(i) Show that $(M_n)_{n\in\mathbb{N}}$ is a martingale with respect to a suitably chosen filtration.

(ii) Use the law of large numbers to show that $(M_n)_{n\in\mathbb{N}}$ converges almost surely and determine the limit.

(iii) Show that for all $a \in \mathbb{R}$ the random variable $T := \min\{k \geq 1 : M_k = a\}$ is a stopping time with respect to the natural filtration $(\mathcal{F}_n)_{n\in\mathbb{N}}$ of $(X_n)_{n\in\mathbb{N}}$, i.e. $\mathcal{F}_n := \sigma(X_1, \ldots, X_n)$ for all $n \in \mathbb{N}$.

Please use the provided blank pages for your solutions!
