TU München
Zentrum Mathematik
Lehrstuhl für Wahrscheinlichkeitstheorie

Probability Theory
Summer Term 2014
Berger/Salvi/Miller

Exercise sheet 1

σ-Algebras, Measurable Maps and Random Variables

Information:

- **Registration:** You will be able to sign up for the exercise classes via TUMonline starting from 8th April 2014, 19:00.
- **Website:** http://www-m14.ma.tum.de/en/teaching/ss14/probability-theory/
- **Bonus system:** Please look up the details of the bonus system for the exams on our website.
- **Homework:** Please hand in your solutions in groups of 2 or 3 in the lecture.

From now on, let Ω be a non-empty set and $\mathcal{F} \subset 2^{\Omega}$ a family of subsets of the power set. Recall the following definitions.

Definition
A system of sets $\mathcal{F}$ is called *algebra* (on Ω), if the following three conditions hold.

(i) $\Omega \in \mathcal{F}$,

(ii) $A, B \in \mathcal{F} \Rightarrow A \cap B \in \mathcal{F}$,

(iii) $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$.

Furthermore, $\mathcal{F}$ is called *σ-algebra*, if it is an algebra and

$$\{A_n\}_{n\geq 1} \subseteq \mathcal{F} \Rightarrow \bigcap_{n=1}^{\infty} A_n \in \mathcal{F}.$$

Definition
Let $(\Omega_1, \mathcal{F}_1)$, $(\Omega_2, \mathcal{F}_2)$ be measurable spaces (Messräume). A map $X : \Omega_1 \mapsto \Omega_2$ is called *measurable*, if for all $A \in \mathcal{F}_2$

$$X^{-1}(A) = \{\omega \in \Omega_1 : X(\omega) \in A\} \in \mathcal{F}_1.$$

If $(\Omega_2, \mathcal{F}_2) = (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$, then X is called *random variable* ($n = 1$) or *random vector* ($n \geq 2$).

Definition
Let $\mathcal{F}$ be an algebra on Ω. A map $\mu : \mathcal{F} \mapsto [0, \infty]$ is called *measure*, if

Please turn the page!

(i) $\mu(\emptyset) = 0$,

(ii) for any pairwise disjoint sets $\{A_n\}_{n\geq 1}$

$$\bigcup_{n=1}^{\infty} A_n \in \mathcal{F} \Rightarrow \mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n). \quad (\sigma\text{-additivity})$$

Homework:
Please hand in the solutions of Exercises 1.1-1.4 in the lecture on 15th of April.

Exercise 1.1 (4 Points)
Let $(\Omega_1, \mathcal{F}_1)$ and $(\Omega_2, \mathcal{F}_2)$ be measurable spaces. In particular, set $\Omega_1 = \Omega_2 = \{1, 2, 3\}$ and $\mathcal{F}_1 = \mathcal{F}_2 = \{\emptyset, \{1\}, \{2, 3\}, \{1, 2, 3\}\}$.

(i) Find a map $X : \Omega_1 \to \Omega_2$ that is measurable.

(ii) Find a map $X : \Omega_1 \to \Omega_2$ that is not measurable.

(iii) How many different maps $X : \Omega_1 \to \Omega_2$ are there? How many of them are measurable?

Exercise 1.2 (4 Points)
Let $(\Omega, \mathcal{F})$ be a measurable space and $X, Y : (\Omega, \mathcal{F}) \mapsto (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ maps. Prove or disprove the following.

(i) If X, Y are random variables, then $X + Y$ is also a random variable.

(ii) If $X + Y$ is a random variable, then X, Y are also random variables.

Hint: Recall the following facts from the Measure Theory course:

- *The Borel σ-algebra on $\mathbb{R}$ is generated by sets of the form $(-\infty, a)$, $a \in \mathbb{R}$;*
- *in order to check the measurability of a map, it is enough to check the 'measurability property' for the generators of the σ-algebra;*
- *all in all you can use the statement of Exercise 1.8.*

Exercise 1.3 (6 Points)

(i) Let $\Omega = \mathbb{R}$. Show that

$$\mathcal{F} := \{A \subset \mathbb{R} : |A| < \infty \text{ or } |A^c| < \infty\}$$

is an algebra on Ω, but not a σ-algebra. With $|A|$ we indicate the cardinality of A (not its Lebesgue measure).

(ii) Find $\sigma(\mathcal{F})$.

Exercise 1.4 (6 Points)
Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. We define a family of sets by

$$\mathcal{A}_\mu := \{A \subseteq \Omega : \exists E, F \in \mathcal{A} \text{ such that } E \subseteq A \subseteq F, \mu(F \setminus E) = 0\}$$

and furthermore, we define a set function $\bar{\mu}$ on $\mathcal{A}_\mu$ by

$$\bar{\mu} : \mathcal{A}_\mu \to [0, \infty],\ B \mapsto \sup\{\mu(A) : A \in \mathcal{A}, A \subseteq B\}.$$

Show that the following assertions hold:

(i) $\mathcal{A}_\mu$ is a σ-algebra on Ω satisfying $\mathcal{A} \subseteq \mathcal{A}_\mu$.

(ii) $\bar{\mu}$ is a measure on $\mathcal{A}_\mu$ which extends μ, i.e., we have $\bar{\mu}(A) = \mu(A)$ for all $A \in \mathcal{A}$.

Remark: $(\Omega, \mathcal{A}_\mu, \bar{\mu})$ is called the completion of the measure space $(\Omega, \mathcal{A}, \mu)$.

Exercises for the tutorial:
Exercises 1.5-1.8 will be discussed in the tutorials between 15th and 25th of April. There will be no tutorial on Wednesday, 23rd of April!

Exercise 1.5
Let $\Omega = \{-2, -2, -1, 0, 1, 2, \text{strawberry ice cream}\}$ and $\varepsilon = \{\{-2, -1\}\}$.

(i) Specify $\sigma(\varepsilon)$.

(ii) Describe all probability measures $\mathbb{P} : \sigma(\varepsilon) \to [0, 1]$.

(iii) Let $(\Omega, \sigma(\varepsilon))$, $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be measurable spaces. Describe all random variables $X : \Omega \to \mathbb{R}$.

Exercise 1.6
Let $(\Omega_1, \mathcal{F}_1)$ and $(\Omega_2, \mathcal{F}_2)$ be measurable spaces. Prove or disprove the following statements.

(i) If $\mathcal{F}_1 = 2^{\Omega_1}$, then every map $X : \Omega_1 \mapsto \Omega_2$ is measurable.

(ii) If $\mathcal{F}_2 = \{\emptyset, \Omega_2\}$, then every map $X : \Omega_1 \mapsto \Omega_2$ is measurable.

(iii) If $\mathcal{F}_1 = \{\emptyset, \Omega_1\}$, $\mathcal{F}_2 \neq \{\emptyset, \Omega_2\}$, then the map $X : \Omega_1 \mapsto \Omega_2$ is measurable iff $X(\omega) \equiv c$ for one $c \in \Omega_2$.

(iv) If $\mathcal{F}_1 = \{\emptyset, \Omega_1\}$, $\Omega_2 = \mathbb{R}$ and $\mathcal{F}_2 = \mathcal{B}(\mathbb{R})$, then the map $X : \Omega_1 \mapsto \Omega_2$ is measurable iff $X(\omega) \equiv c$ for one $c \in \mathbb{R}$.

Exercise 1.7
Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $(A_n)_{n\in\mathbb{N}_0}$ be a sequence in $\mathcal{F}$.

(i) Prove the sub-additivity property $\mathbb{P}\left(\bigcup_{n=0}^{\infty} A_n\right) \leq \sum_{n=0}^{\infty} \mathbb{P}(A_n)$.

(ii) If $A_0 \subseteq A_1 \subseteq A_2 \subseteq \ldots$, then $\mathbb{P}\left(\bigcup_{n=0}^{\infty} A_n\right) = \lim_{n\to\infty} \mathbb{P}(A_n)$.

(iii) If $A_0 \supseteq A_1 \supseteq A_2 \supseteq \ldots$, then $\mathbb{P}\left(\bigcap_{n=0}^{\infty} A_n\right) = \lim_{n\to\infty} \mathbb{P}(A_n)$.

Exercise 1.8

Let $(\Omega, \mathcal{F})$ be a measurable space and $X : \Omega \mapsto \mathbb{R}$ a map. Prove that X is $\mathcal{F} - \mathcal{B}(\mathbb{R})$-measurable iff for all $q \in \mathbb{Q}$ we have $\{\omega \in \Omega : X(\omega) < q\} \in \mathcal{F}$.