TU München
Zentrum Mathematik
Lehrstuhl für Wahrscheinlichkeitstheorie

Probability Theory
Summer Term 2014
Berger/Salvi/Miller

Exercise sheet 10

Optional Stopping

Homework:
Please hand in the solutions of Exercises 10.1-10.4 in the lecture on 24th of June.

Exercise 10.1 (5 Points)
Let $(X_n)_n$ be a submartingale w.r.t a filtration $(\mathcal{F}_n)_n$ and T, S stopping times such that $\mathbb{P}(T \leq S \leq C) = 1$ for some constant C. Show that $\mathsf{E}[X_T] \leq \mathsf{E}[X_S]$.

Exercise 10.2 (5 Points)

Let $X_1, X_2, ...$ be i.i.d. Bernoulli random variables with parameter $p \in (0, 1)$. Let

$$Z_n := c_n \prod_{k=1}^{n} X_k.$$

(i) Find c_n that makes Z_n a martingale and prove that Z_n is indeed a martingale in this case.

(ii) Let $\tau := \inf\{n > 0 : \ Z_n = 0\}$. Prove or disprove that τ is a stopping time w.r.t. the natural filtration and calculate its distribution.

(iii) Prove that $Z_n \to 0$ a.s..

Exercise 10.3 (5 Points)
Flip a sequence of i.i.d. fair coins such that each coin gets a head(H) (or a tail(T)) with probability 1/2. What is the expectation of the first time that we get "HTH"?

Exercise 10.4 (5 Points)
Let $(X_i)_{i=1}^{\infty}$ be iid random variables with $E|X_1| < \infty$ and let $S_n := X_1 + \cdots + X_n$. If T is a stopping time with $ET < \infty$, show that $ES_T = E[X_1]E[T]$.

Exercises for the tutorial:
Exercises 10.5-10.8 will be discussed in the tutorials between 24th and 27th of June.

Exercise 10.5
Let (X_n) be adapted and integrable. Show that (X_n) is a martingale iff $\mathsf{E}[X_T] = \mathsf{E}[X_1]$ for all bounded stopping times T.

Please turn the page!

Exercise 10.6
Let $(X_n)_n$ be a simple random walk on $\{0, 1, 2, 3\}$. Define the stopping times $T = \inf\{n \geq 0 : X_n \in \{0, 3\}\}$. Calculate $\tau_i = \mathsf{E}[T|X_0 = i]$ and $\nu_i = \mathsf{P}[X_T = 0|X_0 = i]$ for $i \in \{0, 1, 2, 3\}$.

Exercise 10.7
Let $(X_n)_n$ be a simple random walk on $\mathbb{Z}$ starting in $X_0 = 0$ and let $a < 0 < b$. Consider the random time

$$T_{a,b} := \inf\{n \geq 0 : X_n \in \{a, b\}\}.$$

(i) Show that

$$\mathsf{P}(X_{T_{a,b}} = a) = \frac{b}{|a| + b} \quad \text{and} \quad \mathsf{P}(X_{T_{a,b}} = b) = \frac{|a|}{|a| + b}.$$

(ii) Show that $\mathsf{E}[T_{a,b}] = |a|b$.

Exercise 10.8
Give an example of a martingale that is not convergent almost surely.