Exercise sheet 11

Optional Stopping and Martingale Convergence

Homework:
Please hand in the solutions of Exercises 11.1-11.4 in the lecture on 1st of July.

Definition Let T be a stopping time. The stopping time σ-algebra $\mathcal{F}_T$ is

$$\mathcal{F}_T = \{A \in \mathcal{F} | A \cap \{T \le n\} \in \mathcal{F}_n \text{ for all } n\},$$

the collection of all events that are measurable up to the random time T. If T is a stopping time, then $\mathcal{F}_T$ is a σ-algebra and X_T is $\mathcal{F}_T$ measurable.

Exercise 11.1 (5 Points)
Let $(X_n)_{n=0}^N$ be a simple random walk with $X_0 = 0$.

(i) Write down carefully a suitable probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

(ii) Let $N = 2$ and $T = 1$ if $X_1 = 1$, $T = 2$ if $X_1 \neq 1$. Show that T is a stopping time.

(iii) Derive $\mathcal{F}_T$ and X_T and the probability distribution of X_T.

(iv) Compute $\mathsf{E}[X_2 | \mathcal{F}_T]$.

Exercise 11.2 (3 Points)
Let (S_n) be a simple random walk starting in $S_0 = 0$. Define the random time $T := \inf\{n \ge 1 : S_n \in \{-1, 0, 4\}\}$. Calculate $\mathsf{P}(S_T = 4)$.

Exercise 11.3 (6 Points)
Let (X_n) be a non-negative supermartingale, T be a stopping time with respect to the same filtration.

(i) Show that the stopped process $(X_{n \wedge T})_n$ is a supermartingale.

(ii) Use Lemma of Fatou to show that $\mathsf{E}[X_T \, \mathrm{I}_{\{T<\infty\}}] \le \mathsf{E}[X_0]$.

(iii) Conclude that for all $a > 0$

$$a\mathsf{P}\left(\sup_{n\ge 1} X_n \ge a\right) \le \mathsf{E}[X_0].$$

Please turn the page!

Remark: Please note that the original statement in this exercise part (ii) was wrong. Points in from exercise parts (ii) and (iii) will be counted as bonus points and will not be included in the total possible points for the exam bonus. See solutions for a counter-example to the original statement.

Exercise 11.4 (6 Points)
Let (X_n) be a martingale and $(\mathcal{F}_n)$ its natural filtration. Show that following claims:

(i) If (X_n) is predictable, i.e. $X_n \in \mathcal{F}_{n-1}$, then (X_n) is constant and $X_n = X_0$ for all $n \in \mathbb{N}$.

(ii) If X_n is independent of $\mathcal{F}_{n-1}$ for all $n \in \mathbb{N}$, then (X_n) is constant and $X_n = \mathsf{E}[X_0]$ for all $n \in \mathbb{N}$.

(iii) For all $n \in \mathbb{N}$ the martingale difference $X_n - X_{n-1}$ is uncorrelated to any integrable $\mathcal{F}_{n-1}$-measurable random variable. Conclude that

$$\mathrm{Var}\,(X_n) = \mathrm{Var}\,(X_0) + \sum_{k=1}^{n} \mathrm{Var}\,(X_k - X_{k-1}).$$

(iv) Let Y be an integrable random variable $\mathcal{G}_n$ some filtration. Define $Z_n = \mathsf{E}[Y|\mathcal{G}_n]$ for all $n \in \mathbb{N}$. Show that (Z_n) is a martingale w.r.t $(\mathcal{G}_n)$.

Exercises for the tutorial:
Exercises 11.5-11.8 will be discussed in the tutorials between 1st and 4th of July.

Exercise 11.5
Let (X_n) be a simple random walk starting in $X_0 = 0$. Define the stopping time $T := \inf\{n \geq 0 : X_n = 1\}$. Show that $\mathsf{E}[T] = \infty$. *Remark: This is no contradiction to the fact that $T < \infty$ a.s.! Can you give the distribution of a random variable Y such that Y has no expectation, but Y is finite a.s.?*

Exercise 11.6
Lat $a, b > 0$, (S_n) be a (non simple) random walk with increments (X_k) having finite moment generating function $m(\theta) = \mathsf{E}[e^{\theta X_1}]$ starting in $S_0 = 0$. Let θ satisfy $m(\theta) \geq 1$.

(i) Prove that for $T = \inf\{k \geq 0 : S_k \notin (-a, b)\}$ we have $\mathsf{E}[m(\theta)^{-T} e^{\theta S_T}] = 1$.

(ii) Let $\mathsf{P}(b) = \mathsf{P}(S_T \geq b)$, $\mathsf{P}(a) = \mathsf{P}(S_T \leq -a)$. Choose $\theta \neq 0$ such that $m(\theta) = 1$ and derive an estimate for $\mathsf{P}(a)$ and $\mathsf{P}(b)$.

Exercise 11.7
Let $(X_n)_{n \in \mathbb{Z}}$ be i.i.d. bounded random variables. Let $S_n = \sum_{k=-n, k \neq 0}^{n} \frac{X_k}{k}$.

(i) Prove that the sequence S_n is a martingale.

(ii) Prove that the sequence S_n converges a.s.

Exercise 11.8
Let $(Y_n)_{n=1}^{\infty}$ be iid random variables with $P(Y_1 = 0) = P(Y_1 = 2) = 0.5$. Prove that $X_n := \prod_{i=1}^{n} Y_i$ is a martingale, but $\lim_{n\to\infty} X_n = 0$, almost surely.
Remark: This is an example that a martingale is convergent almost surly but not in L^1.