Exercise sheet 12
Repetition

Homework: There is no more homework on this sheet.

Question and Repetition Sessions:
Short solutions to the exercises on this sheet will be posted online a week before the exam. You are not required to know the results and solutions on this sheet. This sheet should be used to practice and see whether you understood everything. Therefore, it is recommended that you try to solve the problems yourself. If you have any questions, you may ask your tutors in the repetition sessions. There will be four question and repetition sessions in the week before the exam. Please go to the website for dates (`http://www-m14.ma.tum.de/lehre/ss14/probability-theory/`).

Exercise 12.1
Let X be a random variable that has a density and set $V = X - \max\{0, X\}$.

(i) Calculate the distribution of V in terms of the distribution of X.

(ii) Find a condition on X that implies that V has a density.

(iii) Find two measures P_{A} and P_{NA} and constants $0 \leq a, b \leq 1$ such that P_{A} is atomic, P_{NA} is non-atomic and the measure of V can be written as $\mathsf{P}_V = a\mathsf{P}_{\mathrm{A}} + b\mathsf{P}_{\mathrm{NA}}$.

(iv) Let X be uniform on $[-1, 1]$. Is P_V absolutely continuous with respect to P_X?

Exercise 12.2
Let X be a random variable and assume that X is independent of itself. Show that there exists a constant a such that $\mathsf{P}(X = a) = 1$.

Exercise 12.3
Let $(Y_n)_{n\in\mathbb{N}}$ be a sequence of i.i.d. random variables on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$ which are uniformly distributed on the interval $[0, 1]$. Further, we define

$$X_n := \min\{Y_1, \dots, Y_n\}.$$

(i) Prove that the cumulative distribution function F_n of X_n satisfies

$$F_n(t) = \left[1 - (1-t)^n\right] \cdot \mathbb{I}_{[0,1]}(t) + \mathbb{I}_{(1,\infty)}(t).$$

(ii) Prove that X_n is integrable and compute $\mathsf{E}[X_n]$.

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(iii) Prove that the series

$$\sum_{n=1}^{\infty} \mathbb{P}(n^\alpha X_n \geq \varepsilon)$$

converges for all $\alpha \in [0,1)$ and $\varepsilon > 0$.
Hint: You can make use of the fact that $\log(1-x) \leq -x$ holds for all $x < 1$.

(iv) Prove that for $\alpha \in [0,1)$ the sequence $(n^\alpha X_n)_{n\in\mathbb{N}}$ converges $\mathbb{P}$-a.s. towards 0.

Exercise 12.4
Prove that if (X_n) converges in L^p for some $p \in \mathbb{N}$, then (X_n) converges in probability.

Exercise 12.5
Let $X_1, X_2, \dots$ be i.i.d., $p > 0$. Show that

$$\frac{X_n}{\sqrt[p]{n}} \xrightarrow{\text{a.s.}} 0 \qquad \Longleftrightarrow \qquad \mathsf{E}|X_1|^p < \infty$$

by proving each implication in the following chain:

$$\begin{aligned} & \frac{X_n}{\sqrt[p]{n}} \xrightarrow{\text{a.s.}} 0 \\ \Rightarrow \quad & \sum_n \mathsf{P}(|X_1|^p > n) < \infty \\ \Rightarrow \quad & \mathsf{E}|X_1|^p < \infty \\ \Rightarrow \quad & \forall \varepsilon > 0: \quad \sum_n \mathsf{P}(|X_1|^p > \varepsilon^p n) < \infty \\ \Rightarrow \quad & \frac{X_n}{\sqrt[p]{n}} \xrightarrow{\text{a.s.}} 0. \end{aligned}$$

Exercise 12.6
Let X, Y be random variables on $\{1, 2, \dots, N\}$ for some $N \in \mathbb{N}$, such that for all $A \in \mathcal{F}$ $\mathsf{E}[X\,\mathrm{I}_A] \leq \mathsf{E}[Y\,\mathrm{I}_A]$. Show that $X \leq Y$ a.s.

Exercise 12.7
Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a probability space. Prove the conditional Markov inequality, i.e. for a positive random variable X and $\mathcal{G} \subseteq \mathcal{F}$ for all $a > 0$

$$\mathsf{P}(X \geq a|\mathcal{G}) \leq \frac{\mathsf{E}[X|\mathcal{G}]}{a}.$$

Exercise 12.8

(i) Let (X_n), (Y_n) be two sequences of random variables such that (X_n) converges to X in distribution, (Y_n) converges to Y in probability and Y is constant a.s., i.e. $\mathsf{P}(Y = c) = 1$ for some $c \in \mathbb{R}$. Show that $X_n + Y_n$ converges to $X + c$ in distribution. *Remark: The statement is not true, if Y is not constant a.s.. Can you find a counterexample?*

(ii) Let (X_n) be i.i.d. random variables with zero mean and finite variance. Define for all $n \in \mathbb{N}$

$$Z_n := \left(1 - \frac{1}{n}\right) X_n.$$

Show that a central limit theorem holds for Z_n, i.e.

$$S_n := \frac{1}{\sqrt{n}} \sum_{k=1}^{n} Z_k$$

converges in distribution and determine the limit.