TU München
Zentrum Mathematik
Lehrstuhl für Wahrscheinlichkeitstheorie

Probability Theory
Summer Term 2014
Berger/Salvi/Miller

Exercise sheet 2

Integration, Fatou's Lemma, Dominated and Monotone Convergence Theorem

Homework:
Please hand in the solutions of Exercises 2.1-2.4 in the lecture on 29th of April.

Exercise 2.1 (10 Points)

(i) Let $f_n \geq 0$, be μ-integrable functions. Then

$$\int \Big(\sum_{n=1}^{\infty} f_n \Big) \mathrm{d}\mu = \sum_{n=1}^{\infty} \int f_n \mathrm{d}\mu.$$

(ii) Let f be a μ-integrable function. Then

$$\lim_{n\to\infty} \int |f| 1_{|f|\geq n} \mathrm{d}\mu = 0.$$

(iii) Show that if f is μ-integrable, then for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\mu(A) < \delta$ implies $\int_A |f| \mathrm{d}\mu < \varepsilon$.

Exercise 2.2 (5 Points)
Let $p > 0$ and $X \geq 0$ a random variable with $\mathbb{E}[X^p] < \infty$. Show that

(i) $y^p \mathbb{P}(X > y) \xrightarrow{y\to\infty} 0$

(ii) $\mathbb{E}[X^p] = p \int_0^\infty y^{p-1} \mathbb{P}(X > y) \mathrm{d}y$.

Exercise 2.3 (5 Points)
Let $(\Omega, \mathcal{F})$ be a measureable space, $A_i \in \mathcal{F}$ for all $i \geq 1$ and $f_n : \Omega \mapsto \mathbb{R}$ for all $n \geq 1$ be a $\mathcal{F} - \mathcal{B}(\mathbb{R})$-measurable functions. Show that

(i) $\{\omega \in \Omega : \overline{\lim}_{n\to\infty} \mathrm{I}_{A_n}(\omega) = 1\} = \{\omega \in \Omega : \sum_{n\to\infty} \mathrm{I}_{A_n}(\omega) = \infty\}$, where $\mathrm{I}_A(\cdot)$ is the indicator function of the set A.

(ii) $A := \{f_n \text{ converges to } c \in \mathbb{R}\} \in \mathcal{F}$. (Hint: Express A^c using $\overline{\lim} f_n$ and $\underline{\lim} f_n$ and remember that they are measurable functions.)

Please turn the page!

Exercise 2.4 (5 Points)
Remember that if $X \sim \mathcal{N}(\mu_X, \sigma_X^2)$ and $Y \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$ are two independent normal random variables, then $X + Y \sim \mathcal{N}(\mu_X + \mu_Y, \sigma_X^2 + \sigma_Y^2)$. Furthermore, for $a \in \mathbb{R}$, $aX \sim \mathcal{N}(a\mu_X, a^2\sigma_X^2)$.
Take a sequence $X_1, X_2, ...$ of i.i.d. normal random variables with $X_1 \sim \mathcal{N}(0,1)$. Prove that, for any $K > 0$,

$$\mathbb{P}\Big(\overline{\lim_{N\to\infty}} \frac{1}{\sqrt{N}} \sum_{i=1}^{N} X_i > K \Big) > 0.$$

Exercises for the tutorial:
Exercises 2.5-2.9 will be discussed in the tutorials between 29th of April and 2nd of May. The tutorial on group 3 would be on 1st of May (public holiday). We will announce a replacement date for this tutorial on the website.

Exercise 2.5
Let μ be the Lebesgue measure on $\mathbb{R}$.

- Give an example of a sequence of μ-measurable functions $f_1, f_2, ...$ on $\mathbb{R}$, with $f_i \geq 0$ for every $i \in \mathbb{N}$, that has a point-wise limit, but such that $\int_{\mathbb{R}} f_n \mathrm{d}\mu$ does not converge.
- Give an example of a sequence of μ-measurable functions $f_1 \leq f_2 \leq \ldots$ such that $\lim_{n\to\infty} f_n$ has a point-wise limit f, $\lim_{n\to\infty} \int f_n$ also has a limit, but

$$\lim_{n\to\infty} \int f_n \mathrm{d}\mu \neq \int f \mathrm{d}\mu.$$

Exercise 2.6
Let $(\Omega, \mathcal{F})$ be a measurable space, $(A_n)_{n\geq 1} \in \mathcal{F}$ and define

$$B_n = \bigcup_{m=n}^{\infty} A_m, \qquad C_n = \bigcap_{m=n}^{\infty} A_m.$$

Then $C_n \subseteq A_n \subseteq B_n$ and the sequences $(B_n)_{n\geq 1}$ and $(C_n)_{n\geq 1}$ are decreasing and increasing respectively. Their limits are

$$B := \lim B_n = \bigcap_{n\geq 1} B_n = \bigcap_{n\geq 1} \bigcup_{m=n}^{\infty} A_m, \qquad C := \lim C_n = \bigcap_{n\geq 1} C_n = \bigcup_{n\geq 1} \bigcap_{m=n}^{\infty} A_m.$$

We write $B = \overline{\lim}_{n\to\infty} A_n$ and $B = \underline{\lim}_{n\to\infty} A_n$. Show that

(i) $B = \{\omega \in \Omega : \omega \in A_n$ for infinitely many values of $n\}$. (We also write $B = \{\omega \in A_n$ infinitely often (i.o.)$\}$),

(ii) $C = \{\omega \in \Omega : \omega \in A_n \text{ for all but finitely many values of } n\}$.

(iii) Assume that $B = C$ and define $A := \lim A_n = B = C$. Show that A is an event, i.e. $A \in \mathcal{F}$.

(iv) Let $\mathbb{P} : \mathcal{F} \mapsto [0, 1]$ be a probability measure. Show that $\mathbb{P}(A_n) = \mathbb{P}(A)$ under the assumptions of part (iii).

Exercise 2.7

(i) Show that if two random variables X and Y have variances, then they have covariances.

(ii) Show that this is not an "if and only if".

Exercise 2.8
Provide an example of two random variables X, Y such that $\mathbb{C}\text{ov}(X, Y) = 0$ but X and Y are not independent.

Exercise 2.9
Find four random variables such that they are not independent, but each three of them are independent.