Exercise sheet 3

Distribution Function and Radon Nikodym Theorem

Homework:
Please hand in the solutions of Exercises 3.1-3.5 in the lecture on 6th of May.

Exercise 3.1 (4 Points)
Let X, Y be two i.i.d. random variables with common distribution function F and density function f. Calculate distribution and density function of $V = \max\{X, Y\}$ and $U = \min\{X, Y\}$.

Exercise 3.2 (4 Points)
Let X, Y be two independent random variables with exponential distributions and parameters $\lambda, \mu > 0$, $\lambda \neq \mu$, respectively. Determine the density function of $X + Y$.

Exercise 3.3 (4 Points)
For each pair of measures μ and ν, specify which one is absolutely continuous with respect to the other and explain why. If a Radon-Nikodym derivative exists, write it explicitly; if it does not, exhibit an event A that has zero measure with respect to one measure but not with respect to the other.

(i) $\mu \sim \exp(1)$ (the law of an exponential random variable with parameter 2) and $\nu \sim \exp(2)$;

(ii) $\mu \sim \mathcal{N}(0, 1)$ (the law of a normal random variable with mean 0 and variance 1) and $\nu \sim \exp(1)$;

(iii) $\mu \sim \mathcal{N}(0, 1)$ and $\nu \sim \text{Bin}(100, \frac{1}{2})$ (the law of a binomial random variable with parameters 100 and $\frac{1}{2}$).

Exercise 3.4 (4 Points)
Let μ, ν be measures on a measurable space $(\Omega, \mathcal{F})$. Assume that ν is a finite measure.

(i) Show that the following conditions are equivalent:

  (1) $\nu \ll \mu$;

  (2) For any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that for all $E \in \mathcal{F}$ we have that $\mu(E) < \delta$ implies $\nu(E) < \varepsilon$.

(ii) Show that (1) does not imply (2) if ν is an infinite measure.

Please turn the page!

Exercise 3.5 (4 Points)
Let P_1, P_2 be two non-atomic probability measures on the same measurable space. Assume that any two random variables are independent under P_1 if and only if they are independent under P_2. Prove that P_1 and P_2 are equivalent.

Exercises for the tutorial:
Exercises 3.6-3.8 will be discussed in the tutorials between 6th of May and 9th of May.

Exercise 3.6

(i) Let μ, ν be two measures on the same measure space such that $\nu \ll \mu$. Assume that μ is σ-finite. Prove or disprove that ν is σ-finite, too.

(ii) Provide two measures μ, ν on the same measurable space such that ν is σ-finite and $\nu \ll \mu$, but the Radon-Nikodym derivative $\frac{\mathrm{d}\nu}{\mathrm{d}\mu}$ does not exist.

Exercise 3.7
Let $(\Omega, \mathcal{F}, P)$ be a probability space. Prove that if Ω is countable, then P is atomic.

Exercise 3.8
Let $X_1, X_2, X_3, \ldots$ be i.i.d. random variables with $\mathbb{P}(X_1 = i) = \frac{1}{10}$ for $i = 0, \ldots, 9$. What is the distribution of $Y = \sum_{i \geq 1} \frac{X_i}{10^i}$?

Exercise 3.9
As in Exercise 3.4, let P_1, P_2 be two non-atomic probability measures on the same measurable space. Assume that any two random variables are independent under P_1 if and only if they are independent under P_2. Show that $P_1 = P_2$.