Exercise sheet 4

Inequalities, Borel-Cantelli Lemma and Kolmogorov 0-1 law

Homework:
Please hand in the solutions of Exercises 4.1-4.4 in the lecture on 13th of May.

Exercise 4.1 (6 Points)

(i) Let X_1, X_2 be i.i.d. positive random variables and assume that $\mathsf{E}[X_1] < \infty$ but $\mathsf{E}[X_1^2] = \infty$. Let $Y = \min\{X_1, X_2\}$. Show that $\mathsf{E}[Y^2] < \infty$.
(Note that X_1 does not need to have a density function)

(ii) Generalize the previous result: Let $X_1, X_2, ..., X_n$, $n \in \mathbb{N}$, be i.i.d. positive random variables and assume that $\mathsf{E}[X_1] < \infty$ but $\mathsf{E}[X_1^2] = \infty$. Let Y be the second biggest of them. Show that $\mathsf{E}[Y^2] < \infty$.

Exercise 4.2 (6 Points)
Let $(X_n)_{n=1}^{\infty}$ be a sequence of independent variables such that for every $n \in \mathbb{N}$,

$$P(X_n > x) = e^{-x}, \forall x \geq 0.$$

Prove that with probability 1,

$$\overline{\lim_{n\to\infty}} \frac{X_n}{\log n} = 1.$$

Exercise 4.3 (4 Points)
Let $X_1, X_2, \ldots$ be i.i.d. random variables with $\mathbb{P}(X_1 = 1) = p$ and $\mathbb{P}(X_1 = -1) = 1 - p$ for some $p \in [0, 1] \setminus \{1/2\}$. Let $S_n = X_1 + \ldots + X_n$ and define the event $A = \{S_n = 0 \text{ for infinitely many } n\}$.

(i) Decide whether A is a tail event.

(ii) Show that $\mathbb{P}(A) \in \{0, 1\}$.

Theorem 1 *Let $(\mathcal{A}_i)_i \in I$ be an independent collection of σ-algebras. For any partition $(J_k)_{k\in K}$ of I into pairwise disjoint subsets the σ-fields $\sigma\left(\bigcup_{i\in J_k} \mathcal{A}_i\right)$ for any $k \in K$ are independent, too.*

Please turn the page!

Exercise 4.4 (4 Points)
Find three σ-algebras $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$ such that the three are pairwise independent such that $\sigma(\mathcal{A}_1 \cup \mathcal{A}_2)$ is not independent of $\mathcal{A}_3$. Why is this no contradiction to the previous theorem?

Exercises for the tutorial:
Exercises 4.5-4.8 will be discussed in the tutorials between 13th of May and 16th of May.

Exercise 4.5
Let $X_1, X_2, \dots$ be real valued random variables. Are the following events in the tail-σ-algebra?

(i) $A = \left\{\overline{\lim}_{n\to\infty} X_n < c\right\}$

(ii) $B = \{\sum_{k=1}^{\infty} |X_k| < \infty\}$

(iii) $C = \{\sum_{k=1}^{\infty} |X_k| < 1000000\}$.

Exercise 4.6
Find a sequence $(A_k)_{k=1}^{\infty}$ of events such that

$$\sum_{k=1}^{\infty} P(A_k) = \infty$$

but $P(\overline{\lim}_{k\to\infty} A_k) = 0$.

Exercise 4.7
Prove that the second Borel-Cantelli Lemma still holds if we replace the independence of the events with the weaker condition of pairwise-independence.
Hint: Define $X_n = \#$events occurred up to n. Show that $Var(X_n) < \mathsf{E}[X_n]$ and use this fact to show that $\mathbb{P}(X_n < \frac{\mathsf{E}[X_n]}{2}) \to 0$.

Exercise 4.8
Suppose that $X_n \xrightarrow{\text{a.s.}} X$ and $Y_n \xrightarrow{\text{a.s.}} Y$ and show that $X_n + Y_n \xrightarrow{\text{a.s.}} X + Y$. Show that the corresponding result holds for convergence in p-norm and in probability, but not in distribution.