Exercise sheet 5
Convergence of random variables

There are many different notations for the types of convergence. We will use the following notation: Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random variables.

(i) If (X_n) converges to X in probability we write $X \xrightarrow{\mathbb{P}} X$.

(ii) If (X_n) converges to X almost surely we write $X \xrightarrow{a.s.} X$.

(iii) If (X_n) converges to X in L^p we write $X \xrightarrow{L^p} X$.

(iv) If (X_n) converges to X in distribution, i.e. it converges weakly, we write $X \xrightarrow{w} X$. You might also find the notation $X \xrightarrow{D} X$.

Homework:
Please hand in the solutions of Exercises 5.1-5.4 in the lecture on 20th of May.

Exercise 5.1 (6 Points)

Let $X_1, X_2, ..., Y_1, Y_2, ..., X$ and Y random variables. Prove or disprove the following

(i) $X_n \xrightarrow{\mathbb{P}} X$, $X_n \xrightarrow{\mathbb{P}} Y \Rightarrow X = Y$ a.s.

(ii) $X_n \xrightarrow{w} X$, $X_n \xrightarrow{w} Y \Rightarrow X = Y$ a.s.

Exercise 5.2 (5 Points)

Let $X_1, X_2, ...$ be i.i.d. random variables with $\mathsf{E}[X_1] = 0$ and let $\beta \in (1,2)$. Show that

$$\frac{X_1 + X_2 + ... + X_n}{n^{\frac{1}{\beta}}} \xrightarrow{a.s.} 0 \Rightarrow \mathsf{E}[|X_1|^\beta] < \infty.$$

Hint: Why is it enough to prove that $\sum_{n=1}^\infty \mathbb{P}(|X_n| > n^{\frac{1}{\beta}}) < \infty$?

Exercise 5.3 (4 Points)
Assume that a sequence of random variables $(X_i)_{i=1}^\infty$ converges to X in probability. If almost surely, for all $i \in \mathbb{N}$, $|X_i| \leq Y$ for some random variable Y with $E[Y] < \infty$, show that

$$\lim_{i\to\infty} EX_i = EX.$$

Please turn the page!

Exercise 5.4 (5 Points)
Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random variables and $c \in \mathbb{R}$ be a constant. Find the relations ($\subseteq$, $\supseteq$ or $=$) between the following sets:

(i) $\{\overline{\lim}_{n\to\infty} X_n < c\}$ and $\overline{\lim}_{n\to\infty}\{X_n < c\}$

(ii) $\{\overline{\lim}_{n\to\infty} X_n < c\}$ and $\underline{\lim}_{n\to\infty}\{X_n < c\}$

(iii) $\{\overline{\lim}_{n\to\infty} X_n \leq c\}$ and $\underline{\lim}_{n\to\infty}\{X_n \leq c\}$

Exercises for the tutorial:
Exercises 5.5-5.8 will be discussed in the tutorials between 20th of May and 23th of May.

Exercise 5.5

Look at the diagram of page 20 in Prof. Berger's script. Find a sequence of r.v.'s $X_1, X_2, ...$ and a r.v. X such that

(i) $X_n \xrightarrow{w} X$ but $X_n \overset{\mathbb{P}}{\not\to} X$;

(ii) $X_n \xrightarrow{\mathbb{P}} X$ but $X_n \overset{a.s.}{\not\longrightarrow} X$;

(iii) $X_n \xrightarrow{\mathbb{P}} X$ but $X_n \overset{L^p}{\not\longrightarrow} X$;

(iv) $X_n \xrightarrow{a.s.} X$ but $X_n \overset{L^p}{\not\longrightarrow} X$;

(v) $X_n \xrightarrow{L^p} X$ but $X_n \overset{a.s.}{\not\longrightarrow} X$.

Exercise 5.6
Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random variables. Prove Theorem 2.16 ii), i.e. that if $X_n \xrightarrow{w} X$ and X is a constant almost surely, then $X_n \xrightarrow{\mathbb{P}} X$.

Exercise 5.7
The weak law of large numbers states that, if $X_1, X_2, \ldots$ are independent and identically distributed random variables with mean μ and standard deviation $\sigma < \infty$ then for any $\varepsilon > 0$ we have

$$\lim_{n\to\infty} \mathbb{P}\left(\left|\frac{1}{n}\sum_{i=1}^{n} X_n - \mu\right| > \varepsilon\right) = 0.$$

(i) Calculate $\mathsf{E}[S_n]$ and $\mathrm{Var}[S_n]$ for $S_n = \frac{1}{n}\sum_{i=1}^{n} X_n$.

(ii) Use Chebychev's inequality to prove the weak law of large numbers.

Exercise 5.8
Let $(X_n)_{n=1}^{\infty}$ be a sequence of i.i.d. bounded random variables (bounded means that there exists M such that $P(|X_1| < M) = 1$). In addition assume that $E(X_1) = 0$. For every n, let

$$S_n = \frac{1}{n} \sum_{k=1}^{n} X_k.$$

(i) Show that $E(S_n^4) \leq \frac{4}{n^2} E(X_1^4)$.

(ii) Show that

$$S_n \underset{\text{a.s.}}{\longrightarrow} E(X_1). \tag{1}$$

(iii) Show that (1) holds also when we remove the requirement that $E(X_1) = 0$.