Exercise sheet 6

Laws of Large Numbers and Characteristic Functions

Homework:
Please hand in the solutions of Exercises 6.1-6.4 in the lecture on 27th of May.

Exercise 6.1 (4 Points)

Let $(X_n)_{n\geq 1}$ ba a sequence of i.i.d. random variables uniformly distributed on the interval $[1,2]$. Show that there exist $c \in \mathbb{R}$ such that

$$\lim_{n\to\infty} \left(\prod_{i=1}^{n} X_i \right)^{\frac{1}{n}} = c \qquad \text{a.s.}$$

and determine the value of c.

Exercise 6.2 (4 Points)

Let $\alpha > 0$ and $(X_n)_{n\geq 1}$ a sequence of i.i.d. uniform random variable on the interval $[0, \alpha]$. For $n \in \mathbb{N}$, let $Y_n := \max\{X_1, ..., X_n\}$ and $Z_n := n(\alpha - Y_n)$. Show that Z_n converges in distribution to a r.v. Z and determine its distribution.

Exercise 6.3 (7 Points)

Let $(X_n)_{n\geq 2}$ be a sequence of independent random variables such that

$$X_n = \begin{cases} n & \text{with prob. } \frac{1}{2n\log n} \\ -n & \text{with prob. } \frac{1}{2n\log n} \\ 0 & \text{with prob. } 1 - \frac{1}{n\log n}. \end{cases}$$

Let $S_n := X_2 + ... + X_{n+1}$.

(i) Prove that $\frac{S_n}{n} \overset{a.s.}{\not\longrightarrow} \mathsf{E}[\frac{S_n}{n}]$ (in a sense, $(X_n)_{n\geq 2}$ does NOT satisfy the strong law of large numbers).

(ii) Prove that, on the other hand, for all $\varepsilon > 0$, $\mathbb{P}(|\frac{S_n}{n} - \mathsf{E}[\frac{S_n}{n}]| > \varepsilon) \to 0$ (in a sense, $(X_n)_{n\geq 2}$ satisfies the weak law of large numbers).

Exercise 6.4 (5 Points)
Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of independent random variables on some probability space

Please turn the page!

$(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathsf{E}[X_n] = 0$ for all $n \in \mathbb{N}$ and $S_n = \sum_{i=1}^n X_i$. Assume furthermore that $\sum_{i=1}^\infty \mathsf{E}[X_i^2] < \infty$. Show that there exists a random variable S such that $S_n \xrightarrow{\mathbb{P}} S$.
(Since the exercise contained a mistake in the first version of the sheet you get +2 BONUS POINTS, if you argue why the exercise is trivial if the random variables are identically distributed.)
Hint: One possibility is to show that S_n is a Cauchy sequence in an appropriate space...

Exercises for the tutorial:
Exercises 6.5-6.8 will be discussed in the tutorials between the 27th and the 30th of May.

Definition
Let $(\mathbb{P}_n)_{n\in\mathbb{N}}$, $\mathbb{P}$ be probability measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. We say that $\mathbb{P}_n \xrightarrow{w} \mathbb{P}$ if for all functions f continous, bounded with compact support

$$\int f \mathrm{d}\mathbb{P}_n \xrightarrow{n\to\infty} \int f \mathrm{d}\mathbb{P}.$$

Remark: If we have random variables $(X_n)_{n\in\mathbb{N}}$ and X with laws $(\mathbb{P}_{X_n})_{n\in\mathbb{N}}$ and $\mathbb{P}_X$ respectively, then $\mathbb{P}_{X_n} \xrightarrow{w} \mathbb{P}_X$ iff $X_n \xrightarrow{w} X$ (See definition in the script, Section 2.4.4).

Theorem 1 (Portmanteau) *Let $(\mathbb{P}_n)_{n\in\mathbb{N}}$, $\mathbb{P}$ be probability measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. The following are equivalent:*

(i) $\mathbb{P}_n \xrightarrow{w} \mathbb{P}$

(ii) $\int f \mathrm{d}\mathbb{P}_n \xrightarrow{n\to\infty} \int f \mathrm{d}\mathbb{P}$ *for all continuous and bounded functions f.*

(iii) $\int f \mathrm{d}\mathbb{P}_n \xrightarrow{n\to\infty} \int f \mathrm{d}\mathbb{P}$ *for all functions uniformly continuous and bounded f.*

(iv) $\overline{\lim}\, \mathbb{P}_n(C) \leq \mathbb{P}(C)$ *for all closed sets C.*

(v) $\underline{\lim}\, \mathbb{P}_n(O) \geq \mathbb{P}(O)$ *for all open sets O.*

(vi) $\lim \mathbb{P}_n(A) = \mathbb{P}(A)$ *for all sets A such that $\mathbb{P}(\partial A) = 0$.*

Exercise 6.5
Prove that (i) holds iff (ii) holds in Theorem 1.

Exercise 6.6
Let $(\mathbb{P}_n)_{n\in\mathbb{N}}$, $\mathbb{P}$ be probability measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ with distribution functions $(F_n)_{n\in\mathbb{N}}$ and F. Then $\mathbb{P}_n \xrightarrow{w} \mathbb{P}$ iff $F_n(x) \to F(x)$ for all $x \in \mathbb{R}$ where $F(x)$ is continuous.

Exercise 6.7
Compute the characteristic functions of the following distributions

(i) Bernoulli distribution, $P(X = 0) = 1 - P(X = 1) = p \in (0, 1)$.

(ii) Exponential distribution with density $f(x) = e^{-\lambda x}, x \in (0, \infty)$, where $\lambda > 0$.

(iii) Geometric distribution $P(X = n) = p(1-p)^n$ for $n \in \{0, 1, \ldots\}$ and parameter $p \in (0, 1)$.

Exercise 6.8

(i) Show that the function $f(t) := e^{t^4}$ is not a characteristic function (*Hint:* Calculate the second derivative of f.)

(ii) Let φ be a characteristic function. Show that also

$$\exp[\varphi - 1] = e^{-1} \sum_{n=0}^{\infty} \frac{\varphi^n}{n!}$$

is a characteristic function.