# Exercise sheet 7
## Characteristic functions and CLT

**Homework:**
Please hand in the solutions of Exercises 7.1-7.4 in the lecture on Tuesday, 3rd of June.

**Exercise 7.1** (6 Points)
For each $n \in \mathbb{N}$, let $X_n$ be a random variable with density

$$f_n(t) := nt^{n-1} 1_{[0,1]}(t).$$

(i) Does the sequence $(X_n)_{n\in\mathbb{N}}$ converge in probability?

(ii) Does the sequence $(X_n)_{n\in\mathbb{N}}$ converge in distribution?

(iii) Consider now the space $([0,1], \mathcal{B}([0,1]), \lambda)$, where $\lambda$ indicates the Lebesgue measure. How can one construct $X_n$ on this space (i.e., give an explicit function $X_n : \omega \to X_n(\omega)$) such that $X_n$ has density $f_n$ and $X_n$ converges Lebesgue-almost surely?

**Exercise 7.2** (6 Points)
Let $X$ be a random variable with characteristic function $\varphi$. We call $X$ lattice-distributed iff there exist $a, b \in \mathbb{R}$ such that $\mathbb{P}(X \in a + b\mathbb{Z}) = 1$. Show that $X$ is lattice distributed iff there exists a $u \neq 0$ such that $|\varphi(u)| = 1$.

**Exercise 7.3** (4 Points)

(i) Calculate the characteristic functions of a binomial random variable $\mathrm{Bin}(n,p)$ and of a Poisson random variable of parameter $\lambda$ ($\mathrm{Pois}(\lambda)$).

(ii) For $k, n = 1, 2, ...$, let $Y_{n,k} \sim \mathrm{Bin}(n, \frac{1}{n2^k})$ be independent random variables and let $Z_n := \sum_{k=1}^n Y_{n,k}$. Show that $Z_n \xrightarrow{(d)} \mathrm{Pois}(1)$.

**Exercise 7.4** (4 Points)
Let $(\lambda_k)_{k\in\mathbb{N}}$ be a positive sequence of numbers such that $\sum_{k\in\mathbb{N}} \lambda_k = 1$ and let $\varphi_k$ be a characteristic function for all $k \in \mathbb{N}$. Show that then $\sum_{k\in\mathbb{N}} \lambda_k \varphi_k(t)$ is also a characteristic function.

**Please turn the page!**

**Exercises for the tutorial:**
Exercises 7.5-7.8 will be discussed in the tutorials between 3rd of June and 6th of June.

**Exercise 7.5**
Let $(X_i)_{i\in\mathbb{N}}$ be a sequence of i.i.d. random variables with mean 0 and variance $\sigma^2 > 0$. Let

$$\varphi_n(\alpha) = \mathbb{P}\Big(\sum_{i=1}^{n} X_n \leq \alpha n\Big).$$

Determine, for every $\alpha \in \mathbb{R}$, $\lim_{n\to\infty} \varphi_n(\alpha)$.

**Exercise 7.6**
Let $(X_n)_{n\in\mathbb{N}}$ be i.i.d. random variables with mean $\mu \in \mathbb{R}$ and variance $\sigma^2 > 0$. For $k \in \mathbb{N}$ define $Y_k := X_k + \frac{1}{2^k}$ and $S_n := Y_1 + ... + Y_n$. Show that the limit

$$\lim_{n\to\infty} \mathbb{P}\Big(\Big|\tfrac{S_n}{n} - \mu\Big| \leq \tfrac{\varepsilon}{\sqrt{n}}\Big)$$

exists and is positive for all $\varepsilon > 0$.

**Exercise 7.7**
Let $X$ and $Y$ be two random variables on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

(i) Prove that if $X$ and $Y$ are independent, then $E[X|Y] = E[X]$ almost surely.

(ii) Give an example that both $X$ and $Y$ are independent of some random variable $Z$, but $E[X|Y,Z] \neq E[X|Y]$.

**Exercise 7.8**
Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

(i) Show that for any random variable $X$ and every $\sigma$-algebra $\mathcal{G} \subseteq \mathcal{F}$: $\mathsf{E}[\mathsf{E}[X|\mathcal{G}]] = \mathsf{E}[X]$.

(ii) Find a random variable $X$ and $\sigma$-algebras $\mathcal{G}_1, \mathcal{G}_2 \subseteq \mathcal{F}$ such that $\mathsf{E}[\mathsf{E}[X|\mathcal{G}_1]|\mathcal{G}_2] \neq \mathsf{E}[\mathsf{E}[X|\mathcal{G}_2]|\mathcal{G}_1]$