Exercise sheet 8

Conditional expectation and Martingales

Homework:
Please hand in the solutions of Exercises 8.1-8.4 until FRIDAY 13th of June, since 10th of June is a public holiday. Either hand them in at the lecture or put them in the letter box in front of lecture room 2.02.01 in Hochbrück, Parkring 11. Sheet 9 will be published on 10th of June and is due on the following Tuesday as usual.

Exercise 8.1 (5 Points)

(i) Let X and Y two i.i.d. random variables. Calculate $\mathsf{E}[X|X+Y] := \mathsf{E}[X|\sigma(X+Y)]$.

(ii) Now let X_1 and X_2 be two independent Poisson random variables with parameters λ_1 and λ_2 respectively. Let $X := X_1$ and $Z := X_1 + X_2$. Calculate $\mathsf{E}[X|Z] = \mathsf{E}[X|\sigma(Z)]$.

Exercise 8.2 (5 Points)
Let $X_1, X_2, ...$ be a sequence of i.i.d. random variables with density function

$$f(t) = \lambda^2 t e^{-\lambda t}, \qquad t \geq 0,$$

for some given $\lambda > 0$. Calculate

$$\lim_{n\to\infty} \mathbb{P}\Big(X_1 + ... + X_n \geq \tfrac{2n}{\lambda}\Big).$$

Exercise 8.3 (5 Points)
A coin shows tails with probability p. Let X_n the number of flips required to see n consecutive tails. Calculate $\mathsf{E}[X_n]$.

Exercise 8.4 (5 Points)
Let X be a $\mathcal{F}$-measurable random variable with $E[X^2] < \infty$. Let $\mathcal{G} \subset \mathcal{F}$. Prove that $E[X|\mathcal{G}] =: Y$ is the 'best guess' of X given the σ-algebra $\mathcal{G}$, in the following sense: For any $\mathcal{G}$-measurable Z,

$$E[(X-Y)^2] \leq E[(X-Z)^2].$$

Show also that if the equality holds, then $Z = Y$, almost surely.
Hint: Make sure that all terms make sense, i.e. show that all expectations are finite!

Please turn the page!

Exercises for the tutorial:
Exercises 8.5-8.8 will be discussed in the tutorials between 11th and 13th of June. The tutorial for group 1 takes place on 12th of June at 10:15 in room 02.13.010 in the mathematics building in Garching.

Exercise 8.5
Let $Z_1, Z_2, \dots$ be standard normal independent random variables and define the σ-algebra by $\mathcal{F}_n = \sigma(Z_1, \dots, Z_2)$. We define for $c \in \mathbb{R}$

$$X_n = \exp\left(\sum_{i=1}^{n}(Z_i - c)\right).$$

(i) Let $c = 1/2$. Show that $(X_n)_n$ is a martingale with respect to $(\mathcal{F}_n)_n$.

(ii) For which $c \in \mathbb{R}$ is $(X_n)_n$ a sub- or supermartingale with respect to $(\mathcal{F}_n)_n$?

Exercise 8.6
Let $(X_n)_n$ and $(Y_n)_n$ be martingales with respect to $(\mathcal{F}_n)_n$ and define

$$Z_n = \begin{cases} X_n + Y_n & \text{if } Y_n > 0, \\ X_n & \text{otherwise.} \end{cases}$$

Is $(Z_n)_n$ a sub-, super- or martingale with respect to $(\mathcal{F}_n)_n$?

Exercise 8.7

(i) Find two martingales whose sum is not a martingale.

(ii) Let f be a convex (concave), continuous and bounded function and $(X_n)_n$ be a martingale. Show that $(f(X_n))_n$ is a sub- (super-) martingale.

Exercise 8.8

(i) Flip a sequence of i.i.d. fair coins such that each coin gets a head(H) (or a tail(T)) with probability 1/2. Let S_1 be the first time that we get an "HT". Show that S_1 is a stopping time with $E[S_1] < \infty$.

(ii) Let $(X_n)_{n=0}^{\infty}$ be a simple random walk, i.e. $X_n = \sum_{k=1}^{n} B_k$ where $(B_n)_n$ is a sequence of i.i.d. random variables that take values ± 1 with probability 1/2. Let $\mathcal{G}_n = \sigma(X_0, \dots, X_n)$. Take $T = \inf\{n < 100 : X_n = \max\{X_k : k = 1, \dots, 99\}\}$. Show that T is not a stopping time.