Exercise sheet 9

More Martingales and Stopping Times

Homework:
Please hand in the solutions of Exercises 9.1-9.4 in the lecture on 17th of June.

Exercise 9.1 (6 Points)
Let $a, b > 0$ and let X, Y be two random variables with values in $\mathbb{Z}_+$ and $\mathbb{R}_+$ respectively, whose distribution is given by the formula

$$\mathbb{P}(X = n, Y \leq t) = b \int_0^t \frac{(ay)^n}{n!} \exp\{-(a+b)y\} \, \mathrm{d}y.$$

Let $n \in \mathbb{Z}_+$ and $h : \mathbb{R}_+ \to \mathbb{R}_+$ be a measurable function; compute $\mathsf{E}[h(Y)|X = n]$ and $\mathsf{E}[Y/(X+1)]$.
Hint: It could be useful to review the Gamma distribution...

Exercise 9.2 (5 Points)
Let $(X_n)_n$ be integrable and adapted to a filtration $(\mathcal{F}_n)_n$, i.e. X_n is $\mathcal{F}_n$-measurable for all $n \in \mathbb{N}$. Find a process $(V_n)_n$ such that V_n is $\mathcal{F}_{n-1}$-measurable for all $n \in \mathbb{N}$, $\mathcal{F}_0 := \{\emptyset, \Omega\}$, and $(X_n - V_n)_n$ is a martingale.

Exercise 9.3 (4 Points)
An urn contains 1 red ball and 1 green ball. At each time we draw one ball out (uniformly from all balls), then put it back with an extra ball of the same color. Let X_n be the fraction of green balls after the n-th draw (I.e, X_n is $\frac{\#\text{red balls}}{\#\text{balls}}$ at time n). Prove that X_n is a martingale.

Exercise 9.4 (5 Points)

Let T be an $(\mathcal{F}_n)_{n \geq 0}$-stopping time (i.e., T is a random variable taking values in $\mathbb{N}_0$ such that the event $\{T \leq n\}$ is $\mathcal{F}_n$-measurable for all $n \in \mathbb{N}_0$) such that for some integer $N > 0$ and $\varepsilon > 0$ it holds

$$\mathbb{P}(T \leq N + n | \mathcal{F}_n) \geq \varepsilon, \quad \text{for every } n \geq 0.$$

Show that $\mathsf{E}[T] < \infty$.
Hint: Find bounds for $\mathbb{P}(T > kN)$*...*

Please turn the page!

Exercises for the tutorial:
Exercises 9.5-9.8 will be discussed in the tutorials between 17th of June and 20th of June. The tutorial of group 3 will take place on Tuesday, 17th of June at 12:15 in room 03.07.023, Seminarraum (5607.03.023), Forschungszentrum.

Exercise 9.5
Let (X_n) be a simple random walk, $X_0 = 0$, and let $\mathcal{G}_n = \sigma(X_1, ..., X_n)$. Prove each of the following claims:

(i) Take $T = \inf\{n : X_n = 9\}$. Then T is a stopping time.

(ii) Take $T = \sup\{n < 100 : X_n = 0\}$. Then T is not a stopping time.

(iii) Take $T = \inf\{n \geq 100 : X_n = \max\{X_k : k = 1, \dots, 99\}$. Then T is a stopping time.

Exercise 9.6

(i) Find an example of a martingale (M_n) and a random time T such that $X_{n \wedge T}$ is not a martingale.

(ii) Let (M_n) be a martingale. Find two stopping times $S \leq T$ with $E[S] < \infty$ such that $E[M_S^2] > E[M_T^2]$.

Exercise 9.7
Let S, T be stopping times of a filtration $(\mathcal{G}_n)_{n=1}^{\infty}$. Prove that $S \vee T, S \wedge T, S + T$ are also stopping times. Here $a \vee b := \max\{a, b\}$ and $a \wedge b := \min\{a, b\}$.

Exercise 9.8
Flip a sequence of iid fair coins such that each coin gets a head(H) (or a tail(T)) with probability 1/2. Let S be the first time that we get an "HT". In Exercise 8.8 a) you have shown that S is a stopping time with $E[S] < \infty$.

(i) What is $E[S]$?

(ii) What is the expectation of the first time that we get "HH"?