Solutions for Exercise sheet 1

σ-Algebras, Measurable Maps and Random Variables

Solution for Exercise 1.1

(i) The identity is measurable.

(ii) Take the map $X : \Omega_1 \to \Omega_2$ such that $X(1) = 1$, $X(2) = 1$ and $X(3) = 2$. X is ot measurable, since the pre-image of the set $\{1\} \in \mathcal{F}_2$ is the set $\{1, 2\} \notin \mathcal{F}_1$.

(iii) There are $3^3 = 27$ different maps between Ω_1 and Ω_2 (for each of the 3 elements in Ω_1 there are 3 possibile images in Ω_2). Only 15 of them are measurable. Keeping $X(1) = 1$, there are five possibilities for the images of 2 and 3:

- $2 \to 1$ and $3 \to 1$;
- $2 \to 2$ and $3 \to 3$;
- $2 \to 3$ and $3 \to 2$;
- $2 \to 2$ and $3 \to 2$;
- $2 \to 3$ and $3 \to 3$.

The same holds when we keep $X(1) = 2$ or $X(1) = 3$, for a total of 15 combinations. All the other maps are not measurable (see e.g., part (ii)).

Solution for Exercise 1.2

(i) If X and Y are random variables, then also $Z = X + Y$ is a random variable.

In fact, we only have to verify (see Hint) that $\{\omega \in \Omega : X(\omega) + Y(\omega) < q\} \in \mathcal{F}$ for all $q \in \mathbb{Q}$. We can rewrite

$$\begin{aligned}\{X + Y < q\} &= \{X < q - Y\} \\ &= \bigcup_{a \in \mathbb{Q}} \big(\{X < a\} \cap \{a < q - Y\}\big).\end{aligned}$$

Since $\{X < a\}$ and $\{a < q - Y\}$ are in $\mathcal{F}$ for each $a \in \mathbb{Q}$ (by measurability of X and Y), also the countable union of the intersection of such events is in $\mathcal{F}$.

(ii) If $X + Y$ is a random variable, X and Y do not have necessarily to be random variables. As an easy counterexample take $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ equipped with the σ-algebra $\mathcal{F} = \{\Omega, \{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}, \emptyset\}$ and the following random variables: $X : \Omega \to \mathbb{R}$ such that $X(\omega_1) = 1$, $X(\omega_2) = 2$, $X(\omega_3) = 1$, $X(\omega_4) = 2$, and $Y : \Omega \to \mathbb{R}$ such that $Y(\omega_1) = -1$, $Y(\omega_2) = -2$, $Y(\omega_3) = -1$, $Y(\omega_4) = -2,$. Then neither X nor Y is a random variable (e.g., $X^{-1}(2) = \{\omega_2, \omega_4\} \notin \mathcal{F}$), but $X + Y \equiv 0$ is constant, hence measurable.

Solution for Exercise 1.3

The first version of the exercise was misunderstandable, sorry for that. $|A|$ could be intended as the cardinality of A or as its Lebesgue measure. This does not make a big difference for the purposes of part (i), but can make things trickyer for part (ii). Apologies.

(i) $\mathcal{F}$ is an algebra:

- $\Omega = \mathbb{R}$ is in $\mathcal{F}$, since $\Omega^c = \emptyset$ is clearly finite.
- If $A, B \in \mathcal{F}$ and are finite, then also $A \cap B$ is finite. If, w.l.o.g., A^c and B are finite, then $A \cap B$ is finite. If both A^c and B^c are finite, then $(A \cap B)^c = A^c \cup B^c$ is also finite.
- If $A \in \mathcal{F}$ is finite, then $(A^c)^c$ is of course finite, so that $A^c \in \mathcal{F}$, too. If on the other hand $A \in \mathcal{F}$ and A^c is finite, then it is clear that $A^c \in \mathcal{F}$.

BUT $\mathcal{F}$ is not a σ-algebra! Take for example $A_1, A_2, ...$ such that $A_i = \{i\}$ for all $i \in \mathbb{N}$. Then clearly $A_i \in \mathcal{F}$ for all i´s, but $A = \bigcup_{i=1}^{\infty} A_i$ is not: both $A = \mathbb{N}$ and $A^c = \mathbb{R} \setminus \mathbb{N}$ are not finite.

(ii) We have to enlarge $\mathcal{F}$ to the smallest σ-algebra containing $\mathcal{F}$. Consider

$$\tilde{F} = \{A \subset \mathbb{R} : \text{ either } A \text{ is countable or } A^c \text{ is countable}\}.$$

We claim that $\tilde{F} = \sigma(\mathcal{F})$.

- $\sigma(\mathcal{F}) \subseteq \tilde{F}$: since clearly $\mathcal{F} \subset \tilde{F}$, all we have to do is to show that $\tilde{F}$ is a σ-algebra.
  * $\mathbb{R} \in \tilde{F}$ is again ok.
  * If $A, B \in \tilde{F}$ and are countable, then also $A \cap B$ is countable. If, w.l.o.g., A^c and B are countable, then $A \cap B$ is countable. If both A^c and B^c are countable, then $(A \cap B)^c = A^c \cup B^c$ is also countable.
  * If $A \in \tilde{F}$ is countable, then $(A^c)^c$ is of course countable, so that $A^c \in \tilde{F}$, too. If on the other hand $A \in \tilde{F}$ and A^c is countable, then it is clear that $A^c \in \tilde{F}$.
  * Finally, the countable union of countable sets is again countable.
- $\tilde{F} \subseteq \sigma(\mathcal{F})$: this is maybe the hardest part. Let $A \in \tilde{F}$ be countable (otherwise we can apply the same way of reasoning to A^c). Then there exists a bijection b between A and $\mathbb{N}$. Let $A_k := b^{-1}(\{1, 2, ..., k\})$ for all $k \in \mathbb{N}$. Note that each A_k is in $\mathcal{F}$ and that $A = \bigcup_{i=1}^{\infty} A_i$. Since, by definition of $\sigma(\mathcal{F})$, $\bigcup_{i=1}^{\infty} A_i \in \sigma(\mathcal{F})$, we have that also $A \in \sigma(\mathcal{F})$.

Solution for Exercise 1.4

(i) Clearly $\mathcal{A} \subseteq \mathcal{A}_\mu$, since every element of $\mathcal{A}$ is also in $\mathcal{A}_\mu$ (take $E = F = A$ in the definition of $\mathcal{A}_\mu$).

We have to prove the four σ-algebra properties:

(i) $\Omega \in A_\mu$ is clear (take $E = F = \Omega$ in the definition of $\mathcal{A}_\mu$).

(ii) Take A_1 and A_2 in $\mathcal{A}_\mu$. We have to prove that $A_1 \cap A_2 \in \mathcal{A}_\mu$.
We know that there exist $E_1, E_2, F_1, F_2 \in \mathcal{A}$ such that: $E_i \subseteq A_i \subseteq F_i$ and $\mu(F_i \setminus E_i) = 0$ for $i = 1, 2$. First observe that $E_1 \cap E_2 \in \mathcal{A}$ and $F_1 \cap F_2 \in \mathcal{A}$. Also, by De Morgan laws, $(F_1 \cap F_2) \setminus (E_1 \cap E_2) \subseteq (F_1 \setminus E_1) \cup (F_2 \setminus E_2)$ so that the l.h.s. has zero μ measure, being a subset of the union of two zero-measure sets. Now, it easy to see that $E_1 \cap E_2 \subseteq A_1 \cap A_2 \subseteq F_1 \cap F_2$, so we are done.

(iii) If $A \in \mathcal{A}_\mu$, with E and F as before, $\mathcal{A} \ni F^c \subseteq A^c \subseteq E^c \in \mathcal{A}$ and $\mu(E^c \setminus F^c) = 0$ (if this was not the case, than there would exist $C \subseteq E^c \setminus F^c$ with $\mu(C) > 0$, and we would arrive to the contradiction $\mu(F \setminus E) \geq \mu(C) > 0$).

(iv) For proving the property for infite intersections, just repeat the argument of part (ii) for infitely many A_i´s.

(ii) $\bar{\mu}(A) = \mu(A)$ for $A \in \mathcal{A}$ is clearly true by the definition of $\bar{\mu}$. We are left to show that $\bar{\mu}$ is indeed a measure on $\mathcal{A}_\mu$.

- $\bar{\mu}(\emptyset) = 0$ is again clear;
- Now take $A_1, A_2, ... \in \mathcal{A}_\mu$ disjoint sets. Let $E_1, E_2, ... \in \mathcal{A}$ and $F_1, F_2, ... \in \mathcal{A}$ be such that $E_i \subseteq A_i \subseteq F_i$ and $\mu(F_i \setminus E_i) = 0$ (i.e., $\mu(E_i) = \mu(F_i)$). Then it follows that $\mu(E_i) = \bar{\mu}(E_i) \leq \bar{\mu}(A_i) \leq \bar{\mu}(F_i) = \mu(F_i)$. At this point it easy to see that, since also the E_i´s have to be disjoint,

$$\sum_{i=1}^{\infty} \bar{\mu}(A_i) = \sum_{i=1}^{\infty} \bar{\mu}(E_i) = \bar{\mu}\Big(\bigcup_{i=1}^{\infty} E_i\Big) = \bar{\mu}\Big(\bigcup_{i=1}^{\infty} A_i\Big).$$