Solutions for Exercise sheet 10
Conditional expectation and Martingales

Solution for Exercise 10.1

Fact: since $(X_n)_n$ is a submartingale, for $A \in \mathcal{F}_k$ we have

$$\mathsf{E}[X_{k+1}1_A] = \mathsf{E}[1_A \mathsf{E}[X_{k+1}|\mathcal{F}_k]] \geq \mathsf{E}[X_k 1_A].$$

We calculate

$$\begin{aligned}\mathsf{E}[X_T] &= \sum_{0\leq i\leq C} \mathsf{E}[X_i 1_{\{T=i\}}] \\ &= \sum_{0\leq i\leq C} \mathsf{E}[X_i 1_{\{T=i,S\geq i\}}] \\ &= \sum_{0\leq i\leq C} \Big(\mathsf{E}[X_i 1_{\{T=S=i\}}] + \mathsf{E}[X_i 1_{\{T=i,S>i\}}]\Big).\end{aligned}$$

The event $\{T = i, S > i\}$ is measurable with respect to $\mathcal{F}_i$ since T and S are stopping times. We can therefore use the Fact for continuing the calculation:

$$\mathsf{E}[X_T] \leq \sum_{0\leq i\leq C} \Big(\mathsf{E}[X_i 1_{\{T=S=i\}}] + \mathsf{E}[X_{i+1} 1_{\{T=i,S=i+1\}}] + \mathsf{E}[X_{i+1} 1_{\{T=i,S>i+1\}}]\Big).$$

We can now iterate the previous step ($\{T = i, S > i + 1\}$ is in fact in $\mathcal{F}_{i+1}$, and so on) to finally obtain

$$\begin{aligned}\mathsf{E}[X_T] &\leq \sum_{0\leq i\leq j\leq C} \mathsf{E}[X_j 1_{\{T=i\leq j=S\}}] \\ &\leq \sum_{0\leq j\leq C} \mathsf{E}[X_j 1_{\{T\leq j=S\}}] \\ &= \mathsf{E}[S].\end{aligned}$$

Solution for Exercise 10.2

(i) $c_n = \frac{1}{p^n}$ seems to be a reasonable choice. In fact

$$\mathsf{E}[Z_{n+1}|\mathcal{F}_n] = \frac{1}{p^{n+1}}\Big(\prod_{j=1}^{n} X_j\Big)\mathsf{E}[X_{n+1}|\mathcal{F}_n] = \frac{1}{p^{n+1}}\Big(\prod_{j=1}^{n} X_j\Big)p = Z_n,$$

where $\mathcal{F}_n = \sigma(X_1, ..., X_n)$. Measurability is here immediate, while integrability derives simply by the fact that $|Z_n| \leq 1$ for all n.

(ii) τ is clearly a stopping time w.r.t. $(\mathcal{F}_n)$. In fact

$$\{\tau = n\} = \{X_1 = 1\} \cap \{X_2 = 1\} \cap ... \cap \{X_{n-1} = 1\} \cap \{X_n = 0\} \in \mathcal{F}_n.$$

In particular, τ is a geometric random variable of parameter $1 - p$, since $\mathbb{P}(\tau = n) = p^{n-1}(1-p)$.

(iii) $\forall \varepsilon > 0$ we have

$$\mathbb{P}(|Z_n| > \varepsilon) \leq \mathbb{P}(Z_n \neq 0) = \mathbb{P}(\tau > n) = p^n.$$

It follows that $\sum \mathbb{P}(|Z_n| > \varepsilon) < \infty$.The first Borel-Cantelli Lemma implies therefore that $\overline{\lim} |Z_n| \leq \varepsilon$ almost surely, for each $\varepsilon > 0$. This implies that $Z_n \to 0$ almost surely.

Solution for Exercise 10.3

Let C_i be the outcome of the i-th coin tossing and $\mathcal{F}_n := \sigma(C_1, \ldots, C_n)$. Let S denote the first time we get 'HTH'. First, since $\{S \leq n\} = \{C_i C_{i+1} C_{i+2} = HTH \text{ for some } i \leq n-2\} \in \mathcal{F}_n$, S is a stopping time. Let $\mathcal{G}_n = \mathcal{F}_{3n}$ and

$$S' = \inf\{n \geq 1 : C_{3n-2} C_{3n-1} C_{3n} = HTH\}.$$

Clearly, $3S' \geq S$ and S' is a stopping time with respect to $(\mathcal{G}_n)$. Note that S' is a geometrical random variable of parameter $1/8$ and therefore $E[S'] < \infty$. So $E[S] < \infty$. We have verified the conditions for applying the optional stopping theorem exactly as for the solution of Exercise 8.8 and we can proceed in the very same way in order to calculate the expected value of S (see Tutorial 8). We obtain

$$0 = E[8 + 2 - S]$$

and $E[S] = 10$.

Solution for Exercise 10.4

First, observe that $\bar{S}_n := S_n - nEX_1$ is a martingale, because it is the sum of n i.i.d. integrable random variables with zero mean. If $X_n \leq M$ are bounded, then $\bar{S}_n$ is a martingale with bounded increment and we can apply optional sampling to conclude that $E[\bar{S}_T] = 0$. So the problem is, how to prove it for unbounded X_n? A natural approach is to use truncation.
For any $M > 0$, by the argument above, $(\sum_{i=1}^n X_i 1_{|X_i| \leq M} - nE[X_1 1_{|X_1| \leq M}])$ is a martingale with bounded increments. Hence by optional sampling,

$$E\left[\sum_{i=1}^{T} X_i 1_{|X_i| \leq M} - TE[X_1 1_{|X_1| \leq M}]\right] = 0.$$

Thus

$$E[\sum_{i=1}^{T} X_i 1_{|X_i|\le M}] = E[T]E[X_1 1_{|X_1|\le M}].$$

Since the left side converges to $E[S_T]$ (One can argue as follows: By monotone convergence, $\lim_{M\to\infty} E[\sum_{i=1}^{T} X_i^+ 1_{|X_i|\le M}] = E[\sum_{i=1}^{T} X_i^+]$ and $\lim_{M\to\infty} E[\sum_{i=1}^{T} X_i^- 1_{|X_i|\le M}] = E[\sum_{i=1}^{T} X_i^-]$. Adding the two equalities yields the limit.) and the right side converges to $E[T]E[X_1]$ (by similar argument) as $M \to \infty$, we obtain

$$E[S_T] = E[T]E[X_1].$$