Solutions for Exercise sheet 11

Conditional expectation and Martingales

Solution for Exercise 11.1

(i) Ω can be chosen to be the space of all the possible N-step paths in $\mathbb{Z}$. To be more formal, we could take for example $\Omega := \{-1,+1\}^N$, interpreting each ± 1 as a step to the right or to the left on the integer line. Since the Ω is finite, we can take $\mathcal{F}$ to be the power set of Ω. The probability $\mathbb{P}$ will assign 1/2 probability to each single $+1$ or -1, so that $\forall \omega \in \Omega$ we have $\mathbb{P}(\omega) = (1/2)^N$. note that with $\mathbb{P}(X_k = a)$, for some $a \in \mathbb{Z}$, we actually mean $\mathbb{P}(\{\omega \in \Omega : \omega_1 + ... + \omega_k = a\})$, where ω_i is the i-th element of the vector ω.

(ii) We consider the following filtration: $\mathcal{F}_1 = \sigma(X_1)$, $\mathcal{F}_2 = \sigma(X_1, X_2)$. We have to show that, for $n = 1, 2$, the event $\{T \le n\}$ is in $\mathcal{F}_n$. But that is straightforward, since $\{T \le 1\} = \{T = 1\} = \{X_1 = 1\} \subset \mathcal{F}_1$, while $\{T \le 2\} = \{X_1 = -1\} \subset \mathcal{F}_1 \subset \mathcal{F}_2$.

(iii) Loosely speaking, if $T = 1$ then $\mathcal{F}_T$ gives us no information on the second step of the random walk, while if $T = 2$ we know everything. To be precise:

$$\mathcal{F}_T = \{\emptyset, \{(+1,+1),(+1,-1)\}, \{(-1,+1)\}, \{(-1,-1)\}, \\ \{(+1,+1),(+1,-1),(-1,+1)\}, \{(+1,+1),(+1,-1),(-1,-1)\}, \Omega\}.$$

For X_T we have $X_T((+1,+1)) = X_T((+1,-1)) = 1$, $X_T((-1,+1)) = 0$ and $X_T((-1,-1)) = -2$. Since each atom has probability 1/4, we have $X_T = 1$ with probability 1/2, $X_T = 0$ with prob. 1/4 and $X_T = -2$ with prob. 1/4.

(iv)

$$\begin{aligned}
\mathsf{E}[X_2|\mathcal{F}_T] &= \mathsf{E}[X_2 - X_T + X_T|\mathcal{F}_T] \\
&= \mathsf{E}[X_2 - X_T|\mathcal{F}_T] + X_T \\
&= \mathsf{E}[(X_2 - X_T)1_{T=1} + (X_2 - X_T)1_{T=2}|\mathcal{F}_T] + X_T \\
&= \mathsf{E}[(X_2 - X_1)1_{T=1} + (X_2 - X_2)1_{T=2}|\mathcal{F}_T] + X_T \\
&= 1_{T=1}\mathsf{E}[X_2 - X_1|\mathcal{F}_T] + X_T \\
&= 1_{T=1}\mathsf{E}[X_2 - X_1] + X_T \\
&= X_T.
\end{aligned}$$

Solution for Exercise 11.2

The easiest thing to do here is probably to condition on the first step of the random walk:

$$\mathbb{P}(S_T = 4) = \mathbb{P}(S_T = 4|S_1 = -1)\mathbb{P}(S_1 = -1) + \mathbb{P}(S_T = 4|X_1 = 1)\mathbb{P}(S_1 = 1).$$

The contribution of the first summand is null, since S_T cannot be equal to 4 if -1 has already been touched. Then we are left to calculate $\mathbb{P}(S_T = 4|X_1 = 1)$. This is equivalent to the probability that a random walk started in 1 hits 4 before 0. We have already calculated this probability in Exercise 10.7, up to a shift, and it equals 1/4. Therefore

$$\mathbb{P}(S_T = 4) = \mathbb{P}(S_T = 4|X_1 = 1)\mathbb{P}(S_1 = 1) = \tfrac{1}{4} \cdot \tfrac{1}{2} = \tfrac{1}{8}.$$

Solution for Exercise 11.3

(i) The integrability of $(X_{n\wedge T})_n$ follows immediately by the integrability of $(X_n)_n$, while the measurability follows by the joint measurability of $(X_n)_n$ and T w.r.t. the filtration we are using. The proof of the supermartingale goes exactly like in Lemma 5.16 of the script.

(ii) The result as stated in the original version of the Exercise sheet (with $E[X_T|T < \infty]$) was wrong. Here an easy counterexample:

Take $(X_n)_n$ as follows:

$$X_n := Y_0 \cdot Y_1 \cdot \ldots \cdot Y_n$$

such that $\mathbb{P}(Y_j = 1/2) = \mathbb{P}(Y_j = 3/2) = 1/2$ for all $j \in \mathbb{N}$, and $X_0 = Y_0 = 1$. Then$(X_n)_n$ is a positive supermartingale (it is a martingale). Consider the stopping time $T := \inf\{n : X_n \geq 2\}$. Notice that T has positive probability of being finite (it has also positive probability of not being finite, but we do not need that). Now

$$2 \leq E[X_T|T < \infty] \not\leq \mathsf{E}[X_0] = 1.$$

Let us solve the new version: By the supermartingale property we have that, for every $n \in \mathbb{N}$,

$$\mathsf{E}[X_{n\wedge T}1_{T<\infty}] \leq \mathsf{E}[X_{n\wedge T}] \leq \mathsf{E}[X_0].$$

On the other hand, Fatou's Lemma (which can be applied because the X_n's are non-negative) tells us that

$$\mathsf{E}[X_T 1_{T<\infty}] = \mathsf{E}[\varliminf_{n\to\infty} X_{n\wedge T}1_{T<\infty}] \leq \varliminf_{n\to\infty} \mathsf{E}[X_{n\wedge T}1_{T<\infty}].$$

Putting together these two statements gives the desired result.

(iii) For any $\varepsilon > 0$, set $T := \inf\{n : X_n \geq a - \varepsilon\}$. This is a stopping time (why?) and it also holds that

$$\mathsf{E}[X_T 1_{T<\infty}] \geq (a - \varepsilon)\mathbb{P}(T < \infty) \geq (a - \varepsilon)\mathbb{P}(\sup_n X_n \geq a),$$

where the last inequality follows from the fact that $\{\sup_n X_n \geq a\} \Rightarrow \{T < \infty\}$. On the left hand side we can apply the result of part (ii) and since the inequality holds true for any $\varepsilon > 0$ we are done.

Note: the annoying game with the ε is meant to avoid the possibility that the $\sup X_n$ was not attained...

Solution for Exercise 11.4

(i) Inductively (all the equalities in the following hold almost surely):

For $n = 1$ we have
$$X_0 = \mathsf{E}[X_1|\mathcal{F}_0] = X_1;$$
the first equality follows from the martingale property, while the second from the predictability of (X_n). Now suppose that $X_{n-1} = X_0$. Then
$$X_0 = X_{n-1} = \mathsf{E}[X_n|\mathcal{F}_{n-1}] = X_n,$$
where we have used again the martingale property for the second equality and predictability for the third.

(ii) For each $n \in \mathbb{N}_0$ we have (using martingale property first and then independence)
$$X_{n+1} = \mathsf{E}[X_{n+2}|\mathcal{F}_n] = \mathsf{E}[X_{n+2}],$$
which implies
$$X_n = \mathsf{E}[X_{n+1}|\mathcal{F}_n] = \mathsf{E}[X_{n+1}] = \mathsf{E}[\mathsf{E}[X_{n+2}]] = \mathsf{E}[X_{n+2}] = X_{n+1}.$$
This argument shows that $(X_n)_n$ is a constant martingale. We also need to show that $X_0 = \mathsf{E}[X_0]$ almost surely (i.e., X_0 is a constant). If it was not the case, X_0 could attain with positive probability two different values a and b (this is true when $(X_n)_n$ takes discrete values, you can easilty adapt this argument for the continuous case...). But since the martingale stays constant for what we have shown, also X_1 should have the value a or b depending on the value of X_0, but then it would not be independent of $\mathcal{F}_0$.

(iii) Let Y be an integrable and $\mathcal{F}_{n-1}$-measurable random variable. Then
$$\mathsf{E}[Y(X_n - X_{n-1})] = \mathsf{E}[\mathsf{E}[Y(X_n - X_{n-1})|\mathcal{F}_{n-1}]] = \mathsf{E}[Y\mathsf{E}[X_n - X_{n-1}|\mathcal{F}_{n-1}]] = 0$$
and
$$\mathsf{E}[Y]\mathsf{E}[X_n - X_{n-1}] = \mathsf{E}[Y]\mathsf{E}[\mathsf{E}[X_n - X_{n-1}|\mathcal{F}_{n-1}]] = 0.$$
The correlation between Y and $X_n - X_{n-1}$ is therefore equal to zero.

Since the variance of uncorrelated variables is the sum of their variances, we have (applying the previous fact with $Y = X_{n-1}$)
$$Var(X_n) = Var(X_n - X_{n-1} + X_{n-1}) = Var(X_n - X_{n-1}) + Var(X_{n-1}).$$
Iterating this formula gives the claim.

(iv) By the definition of conditional expectation, Z_n is measurable with respect to $\mathcal{G}_n$ for every $n \in \mathbb{N}$. The integrability of Z_n follows from Jensen inequality for the absolute value, the tower property and the integrability of Y. We are left to show the martingale property:

$$\mathsf{E}[Z_n|\mathcal{G}_{n-1}] = \mathsf{E}[\mathsf{E}[Y|\mathcal{G}_n]|\mathcal{G}_{n-1}] = \mathsf{E}[Y|\mathcal{G}_{n-1}] = Z_{n-1},$$

where we have used the tower property.

Note: this kind of martingales are sometimes referred to as Doob's Martingales.