Solutions for Exercise sheet 2

Integration, Fatou's Lemma, Dominated and Monotone Convergence Theorem

Solution for Exercise 2.1

(i) Call $g_N := \sum_{n=1}^{N} f_n$. Note that $g_1, g_2, ...$ is a sequence of positive, pointwise increasing functions. Therefore

$$\begin{aligned}\int \Big(\sum_{n=1}^{\infty} f_n\Big)\mathrm{d}\mu &= \int \Big(\lim_{N\to\infty}\sum_{n=1}^{N} f_n\Big)\mathrm{d}\mu \\ &= \int \Big(\lim_{N\to\infty} g_N\Big)\mathrm{d}\mu \\ * &= \lim_{N\to\infty}\int g_N \mathrm{d}\mu \\ &= \lim_{N\to\infty}\int \Big(\sum_{n=1}^{N} f_n\Big)\mathrm{d}\mu \\ &= \lim_{N\to\infty}\sum_{n=1}^{N}\int f_n \mathrm{d}\mu \\ &= \sum_{n=1}^{\infty}\int f_n \mathrm{d}\mu,\end{aligned}$$

where in $*$ we used Monotone Convergence Theorem.

(ii) The functions $g_n := |f|1_{|f|\geq n}$ are clearly dominated by the non-negative function $g := |f|$, i.e. $g_n(x) \leq g(x)$ for (almost) all $x \in \mathbb{R}$. The pointwise limit of g_n is the 0 function. Furthermore, g is integrable, since f is integrable. We can therefore use the Dominated Convergence Theorem to "bring the limit under the integral":

$$\begin{aligned}\lim_{n\to\infty}\int |f|1_{|f|\geq n}\mathrm{d}\mu &= \lim_{n\to\infty}\int g_n \mathrm{d}\mu \\ &= \int \lim_{n\to\infty} g_n \mathrm{d}\mu \\ &= \int 0 \mathrm{d}\mu = 0\end{aligned}$$

(iii) By Part (ii), for any $\varepsilon > 0$, there exists an integer $N = N(\varepsilon)$ such that

$$\int |f|1_{|f|\geq N}\mathrm{d}\mu < \frac{\varepsilon}{2}.$$

Take $\delta = \frac{\varepsilon}{2N}$. Then for any A with $\mu(A) < \delta$,

$$\int_A |f| 1_{|f|<N} \mathrm{d}\mu \leq N \int_A 1_{|f|<N} \mathrm{d}\mu \leq N\mu(A) < \frac{\varepsilon}{2}.$$

Therefore,

$$\int_A |f| \mathrm{d}\mu \leq \int |f| 1_{|f|\geq N} \mathrm{d}\mu + \int_A |f| 1_{|f|<N} \mathrm{d}\mu < \varepsilon.$$

Solution for Exercise 2.2

(i) First of all note that under the indicator function $1\{X > y\}$ we clearly have that $y^p \leq X^p$. We have

$$y^p \mathbb{P}(X > y) \leq \int X^p 1_{\{X>y\}} \mathrm{d}\mathbb{P}.$$

We pass to the limit for $y \to \infty$ and note that we can use Dominated Convergence Theorem, where the dominating function is X^p itself, that we know to be integrable. So

$$y^p \mathbb{P}(X > y) \xrightarrow{y\to\infty} \int \Big(\lim_{y\to\infty} X^p 1_{\{X>y\}} \Big) \mathrm{d}\mathbb{P} = \int 0 \mathrm{d}\mathbb{P} = 0.$$

(ii)

$$\begin{aligned} p \int_0^\infty y^{p-1} \mathbb{P}(X > y) \mathrm{d}y &= p \int_0^\infty y^{p-1} \Big(\int_\Omega 1_{(y,\infty)}(s) \mathbb{P}(X^{-1}(\mathrm{d}s)) \Big) \mathrm{d}y \\ &= \int_0^\infty \Big(\int_\Omega p y^{p-1} 1_{(y,\infty)}(s) \mathbb{P}(X^{-1}(\mathrm{d}s)) \Big) \mathrm{d}y. \end{aligned}$$

Here we can use Fubini's Theorem to change the order of the integrals (notice that the integrand is positive):

$$\begin{aligned} \int_\Omega \Big(\int_0^\infty p y^{p-1} 1_{(y,\infty)}(s) \mathrm{d}y \Big) \mathbb{P}(X^{-1}(\mathrm{d}s)) &= \int_\Omega \Big(\int_0^s p y^{p-1} \mathrm{d}y \Big) \mathbb{P}(X^{-1}(\mathrm{d}s)) \\ &= \int_\Omega s^p \mathbb{P}(X^{-1}(\mathrm{d}s)) \\ &= \mathsf{E}[X^p]. \end{aligned}$$

Solution for Exercise 2.3

(i) "$\subset$ ": If $\omega \in \{\omega \in \Omega : \overline{\lim}_{n\to\infty} \mathrm{I}_{A_n}(\omega) = 1\}$, then we know that $\mathrm{I}_{A_n}(\omega)$ is going to be equal to 1 for infinitely many $n \in \mathbb{N}$, so that the sum of the $\mathrm{I}_{A_n}(\omega)$ has to be infinite and therefore $\omega \in \{\omega \in \Omega : \sum_{n\to\infty} \mathrm{I}_{A_n}(\omega) = \infty\}$, too.

"$\supset$": Conversely, if the sum of the indicator functions is infinite, then it means that $\mathrm{I}_{A_n}(\omega)$ has to be equal to 1 for infinitely many indexes $n \in \mathbb{N}$.

(ii)

$$A = \{f_n \text{ converges to } c \in \mathbb{R}\}^c = \{\underline{\lim}_{n\to\infty} f_n < \overline{\lim}_{n\to\infty} f_n\}$$
$$= \bigcup_{q\in\mathbb{Q}} (\{\underline{\lim}_{n\to\infty} f_n < q\} \cap \{q < \overline{\lim}_{n\to\infty} f_n\}).$$

From the hint one can deduce that $A \in \mathcal{F}$, since it is a countable union of intersections of measurable sets.

Solution for Exercise 2.4

Call $Y_N := \frac{1}{\sqrt{N}} \sum_{i=1}^N X_i$ for all $N \in \mathbb{N}$.

$$\mathbb{P}\Big(\overline{\lim}_{N\to\infty} Y_N > K\Big) = \int_{\{\omega : \overline{\lim}_{N\to\infty} Y_N(\omega) > K\}} \mathrm{d}\mathbb{P}(\omega)$$
$$= \int_\Omega 1_{\{\omega : \overline{\lim}_{N\to\infty} Y_N(\omega) > K\}} \mathrm{d}\mathbb{P}(\omega).$$

Now take $\varepsilon > 0$. We have

$$1_{\{\overline{\lim}_{N\to\infty} Y_N > K\}}(\omega) \geq \begin{cases} 1 & \text{if } Y_N(\omega) > K + \varepsilon \text{ for infinitely many } N \\ 0 & \text{otherwise} \end{cases} = \overline{\lim}_{N\to\infty} 1_{\{Y_N > K+\varepsilon\}}(\omega).$$

Hence, by the Lemma of Fatou,

$$\int_\Omega 1_{\{\omega : \overline{\lim}_{N\to\infty} Y_N(\omega) > K\}} \mathrm{d}\mathbb{P}(\omega) \geq \int_\Omega \overline{\lim}_{N\to\infty} 1_{\{\omega : Y_N(\omega) > K+\varepsilon\}} \mathrm{d}\mathbb{P}(\omega)$$
$$\geq \overline{\lim}_{N\to\infty} \int_\Omega 1_{\{\omega : Y_N(\omega) > K+\varepsilon\}} \mathrm{d}\mathbb{P}(\omega)$$
$$= \overline{\lim}_{N\to\infty} \mathbb{P}(Y_N > K + \varepsilon).$$

But now, thanks to the hint, it is easy to see that $Y_N \sim \mathcal{N}(0,1)$, and therefore (for ε small enough) $\mathbb{P}(Y_N > K + \varepsilon) = C$, where $C > 0$ is a constant independent of N, so that we are done.

BEWARE!
The old (wrong) solution said:

$$1_{\{\overline{\lim}_{N\to\infty} Y_N > K\}}(\omega) = \begin{cases} 1 & \text{if } Y_N(\omega) > K \text{ for infinitely many } N \\ 0 & \text{otherwise} \end{cases} = \overline{\lim}_{N\to\infty} 1_{\{Y_N > K\}}(\omega).$$

This is not true! The first equality should in fact have been a "$\leq$"(just consider the deterministic sequence $Y_N = K + \frac{1}{N}$).