Solutions for Exercise sheet 3

Distribution Function and Radon Nikodyn Theorem

Solution for Exercise 3.1

For each $a \in \mathbb{R}$

$$\begin{aligned} F_V(a) = \mathbb{P}(V < a) &= \mathbb{P}(X < a \text{ and } Y < a) \\ &= \mathbb{P}(X < a)^2 = F(a)^2. \end{aligned}$$

The density of V is therefore given by

$$f_V(a) = F_V'(a) = 2f(a)F(a).$$

Analogously

$$\begin{aligned} 1 - F_U(a) = \mathbb{P}(U > a) &= \mathbb{P}(X > a \text{ and } Y > a) \\ &= \mathbb{P}(X > a)^2 = (1 - F(a))^2. \end{aligned}$$

The density of U is therefore given by

$$f_U(a) = F_U'(a) = (1 - (1 - F(a))^2)' = 2(1 - F(a))f(a).$$

Solution for Exercise 3.2

We start by computing the distribution function of $Z = X + Y$. Take $a \in \mathbb{R}^+$, then

$$\begin{aligned} F_Z(a) = \mathbb{P}(Z < a) &= \int_0^a \mathbb{P}(Y < a - x)\lambda e^{-\lambda x} dx \\ &= \int_0^a \left(1 - e^{-\mu(a-x)}\right)\lambda e^{-\lambda x} dx \\ &= 1 - e^{-\lambda a} - \lambda e^{-\mu a}\left[-\frac{e^{-(\lambda-\mu)x}}{\lambda - \mu}\right]_0^a \\ &= 1 + \frac{\mu e^{-\lambda a}}{\lambda - \mu} - \frac{\lambda e^{-\mu a}}{\lambda - \mu}. \end{aligned}$$

Differenciating the distribution function we obtain the density of $Z = X + Y$:

$$f_{X+Y}(z) = f_Z(z) = F_Z'|_z = \frac{\mu\lambda}{\lambda - \mu}\Big(e^{-\mu z} - e^{-\lambda z}\Big).$$

Solution for Exercise 3.3

(i) μ and ν are equivalent (i.e., $\nu \ll \mu$ and $\mu \ll \nu$). Take a set $A \in \mathcal{B}(\mathbb{R})$. Then,

$$\begin{aligned}\mu(A) &= \int_A \mathrm{d}\mu(x) \\ &= \int_A \mathrm{e}^{-x}\mathrm{d}x \\ &= \int_A 2\tfrac{1}{2}\mathrm{e}^{-x}\mathrm{e}^{-x}\mathrm{e}^{x}\mathrm{d}x \\ &= \int_A \tfrac{1}{2}\mathrm{e}^{x}\mathrm{d}\nu(x).\end{aligned}$$

Therefore $\frac{\mathrm{d}\mu}{\mathrm{d}\nu}(x) = \frac{1}{2}\mathrm{e}^{x}$. Analogously, $\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x) = 2\mathrm{e}^{-x}$.

(ii) In this case $\nu \ll \mu$ but $\mu \not\ll \nu$. To see the first part we proceed as in (i): Take a set $A \in \mathcal{B}(\mathbb{R})$. Then,

$$\begin{aligned}\nu(A) &= \int_A \mathrm{d}\nu(x) \\ &= \int_A \mathrm{e}^{-x}1_{[0,\infty)}(x)\mathrm{d}x \\ &= \int_A \mathrm{e}^{-x}1_{[0,\infty)}(x)\frac{\sqrt{2\pi}}{\sqrt{2\pi}}\mathrm{e}^{-\frac{x^2}{2}+\frac{x^2}{2}}\mathrm{d}x \\ &= \int_A \mathrm{e}^{-x}1_{[0,\infty)}(x)\sqrt{2\pi}\mathrm{e}^{\frac{x^2}{2}}\mathrm{d}\mu(x),\end{aligned}$$

so that $\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x) = \mathrm{e}^{-x}1_{[0,\infty)}(x)\sqrt{2\pi}\mathrm{e}^{\frac{x^2}{2}}$.

On the other hand, if we take an interval $[a, b] \subset \mathbb{R}^-$, with $a < b$, then clearly $\mu(A) > 0$, while $\nu(A) = 0$ (in fact, the exponential random variable can take only positive values).

(iii) μ and ν are this time mutually singular. A short explaination is that ν is a purely atomic measure, while μ is non-atomic. More explicitly, if we take an interval $[a, b] \subset \mathbb{R}^-$, with $a < b$, then clearly $\mu(A) > 0$, while $\nu(A) = 0$ (the binomial takes only positive values). But note that also any set of the type $[a, b]$, with $n < a < b < n + 1$ where $n \in \mathbb{N}$ (n can also be smaller than 100), would do the job (why?).

On the other hand, any singleton $A = \{n\}$, with $n \in \{0, 1, ..., 100\}$, is such that $\nu(A) > 0$ but $\mu(A) = 0$.

Solution for Exercise 3.4

(i) "$\Leftarrow$" Assume (2). Then, if $\mu(E) = 0$, we have $\nu(E) < \varepsilon$ for all $\varepsilon > 0$. Hence $\mu(E) = 0$ and $\nu \ll \mu$.

"$\Rightarrow$" Assume (1). If (2) fails, there exists $\varepsilon > 0$ such that there are sets $(E_n)_{n\in\mathbb{N}}$ with $\mu(E_n) < 2^{-n}$ but $\nu(E_n) \geq \varepsilon$. Hence

$$\mu(\bigcap_{n=1}^{\infty}\bigcup_{i=n}^{\infty} E_i) = \lim_{n\to\infty} \mu(\bigcup_{i=n}^{\infty} E_i) = 0.$$

However,

$$\nu(\bigcap_{n=1}^{\infty}\bigcup_{i=n}^{\infty} E_i) = \lim_{n\to\infty} \nu(\bigcup_{i=n}^{\infty} E_i) \geq \varepsilon,$$

which contradicts (1).

(ii) A possible counterexample is the following: Let μ be the Lebesgue measure on $\mathbb{R}$ and ν such that

$$\nu(E) = \int_E t^{-1}\mathrm{d}t.$$

Solution for Exercise 3.5

Let $\mathcal{A}_1 = \{N \in \mathcal{F} : P_1(N) = 0\}$ be the collection of null-sets under P_1. Then for any $A \in \mathcal{A}_1$, A is independent of itself under P_1, since $P_1(A \cap A) = 0 = P_1(A)P_1(A)$. Hence A is also independent of itself under P_2, i.e., $P_2(A) = P_2(A)P_2(A)$. Thus

$$P_2(A) \in \{0, 1\}.$$

Moreover, we can show that $P_2(A) = 0$. Otherwise, $P_2(A) = 1$, and (by Theorem 1.31 in Prof. Berger's lecture note) we can find a subset $B \subset A$ such that $P_2(B) = 1/2$. Noting that $P_1(B) = 0$, which implies that B and B are independent under P_1, and so also under P_2, we get

$$P_2(B) = P_2(B)P_2(B),$$

which is absurd. So $P_1(A) = 0$ implies $P_2(A) = 0$. We have proved that $P_2 \ll P_1$. Similarly, $P_1 \ll P_2$ and hence $P_1 \sim P_2$.