Solutions for Exercise sheet 4

Inequalities, Borel-Cantelli Lemma and Kolmogorov 0-1 law

Solution for Exercise 4.1

(i)

$$\begin{aligned}
\mathsf{E}[Y^2] &= \int_0^\infty \mathbb{P}(Y^2 > t)\,\mathrm{d}t \\
&= \int_0^\infty \mathbb{P}(Y > \sqrt{t})\,\mathrm{d}t \\
&= \int_0^\infty \mathbb{P}(X_1 > \sqrt{t})^2\,\mathrm{d}t \\
&= 1 + \int_1^\infty \mathbb{P}(X_1 > \sqrt{t})^2\,\mathrm{d}t \\
&\leq 1 + \int_1^\infty \mathbb{P}(X_1 > \sqrt{t})\frac{\mathsf{E}[X_1]}{\sqrt{t}}\,\mathrm{d}t \\
&\leq 1 + \mathsf{E}[X_1]\int_1^\infty \mathbb{P}(X_1 > \sqrt{t})\,\mathrm{d}t \\
&\leq 1 + \mathsf{E}[X_1]^2.
\end{aligned}$$

where in the third line we used the fact that $Y > x$ iff both X_1 and X_2 are $> x$, and in the fifth line we used Markov inequality.

(ii) Notice that $\{Y^2 > t\} = \{\exists X_i, X_j : X_i > t \text{ and } X_j > t\}$. Considering all the combinations that we have for choosing the two indexes i and j, we get:

$$\mathbb{P}(Y^2 > t) \leq \binom{n}{2}\mathbb{P}(X_1 > t,\, X_2 > t) = const.\, P(X_1 > t)^2.$$

The required solution follows then as in (i).

Solution for Exercise 4.2

For any $\varepsilon \in (0,1)$,

$$P(X_n > (1 \pm \varepsilon)\log n) = n^{-(1\pm\varepsilon)}.$$

Hence

$$\sum_{n=1}^{\infty} P(X_n > (1+\varepsilon)\log n) = \sum_{n=1}^{\infty} n^{-(1+\varepsilon)} < \infty$$

and

$$\sum_{n=1}^{\infty} P(X_n > (1-\varepsilon)\log n) = \sum_{n=1}^{\infty} n^{-(1-\varepsilon)} = \infty.$$

By the first and the second Borel-Cantelli lemmas (we can apply the second Borel-Cantelli because X_i's are independent), we conclude that, almost surely,

$$\overline{\lim_{n\to\infty}} \frac{X_n}{\log n} \leq 1+\varepsilon$$

and

$$\overline{\lim_{n\to\infty}} \frac{X_n}{\log n} \geq 1-\varepsilon.$$

Since ε is arbitrary, we have

$$\overline{\lim_{n\to\infty}} \frac{X_n}{\log n} = 1.$$

Solution for Exercise 4.3

(i) A is NOT a tail-event. Losely speaking, it could be important to know the first n outcomes (for a fix $n \in \mathbb{N}$) in order to know whether A happens or not. As an example take the sequence $\omega = (+1, -1, +1, -1, +1, -1, ...)$. Clearly $\omega \in A$. But if we take $\tilde{\omega} = (+1, +1, +1, -1, +1, -1, +1, -1, ...)$, we have $\tilde{\omega} \notin A$. Observe that ω and $\tilde{\omega}$ agree from the third coordinate on, so that it is clear that $A \notin \sigma(X_3, X_4, ...)$.

(ii) Because of (a), it is not possible to use Kolmogorov 0-1 law, in this case. We show that, nevertheless, $\mathbb{P}(A) = 0$.

$$\mathbb{P}(S_{2n} = 0) = \binom{2n}{n} p^n (1-p)^n = (4p(1-p))^n \binom{2n}{n} \frac{1}{2}^n \frac{1}{2}^n \leq (4p(1-p))^n,$$

where we have used the fact that $\binom{2n}{n}\frac{1}{2}^n\frac{1}{2}^n \leq 1$, being just the probability of a particular outcome of a $Binom(2n, \frac{1}{2})$. Also note that $(4p(1-p)) < 1$.

We are now ready to use the first Borel-Cantelli Lemma:

$$\sum_{n\in\mathbb{N}} \mathbb{P}(S_n = 0) = \sum_{n\in\mathbb{N}} \mathbb{P}(S_{2n} = 0) \leq \sum_{n\in\mathbb{N}} (4p(1-p))^n < \infty,$$

hence $\mathbb{P}(A) = 0$.

Solution for Exercise 4.4

The statement is not in contrast with the theorem, since the general independence has been substituted with pairwise independence. Here is a counterexample: Toss 3 independent fair coins and consider the events $A_1 =$"The first and the second coin have the show the same side", $A_2 =$"The second and the third coin have the show the same side",

A_3 =”The first and the third coin have the show the same side”. A_1, A_2 and A_3 are pairwise independent ($\frac{1}{4} = \mathbb{P}(A_1)\mathbb{P}(A_2) = \mathbb{P}(A_1 \cap A_2) = \mathbb{P}(\text{all the coins show the same side}) = \frac{1}{4}$). Now consider, for $i = 1, 2, 3$, the sigma algebras $\mathcal{A}_i = \sigma(A_i)$. Clearly the $\mathcal{A}_i$'s are pairwise independent σ-algebras. But we claim that $\mathcal{A}_1$ is not independent from $\sigma(\mathcal{A}_2 \cup \mathcal{A}_3)$. In fact, the latter σ-algebra contains the event $B = A_2 \cup A_3$, but A_1 and B are not independent:

$$\frac{1}{4} = \mathbb{P}(A_1 \cup B) \neq \mathbb{P}(A_1)\mathbb{P}(B) = \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}.$$