Solutions for Exercise sheet 5

Convergence of random variables

Solution for Exercise 5.1

(i) For every $\varepsilon > 0$ we have

$$\begin{aligned}\mathbb{P}(|X-Y|>\varepsilon) &= \mathbb{P}(|X-X_n+X_n-Y|>\varepsilon)\\ &\le \mathbb{P}(|X-X_n|>\tfrac{\varepsilon}{2})+\mathbb{P}(|Y-X_n|>\tfrac{\varepsilon}{2}) \xrightarrow{n\to\infty} 0\end{aligned}$$

because of the convergence of X_n to X and Y in probability. We have obtained

$$\mathbb{P}(|X-Y|>\varepsilon)=0.$$

It follows: $\mathbb{P}(X\neq Y)=\mathbb{P}\Big(\bigcup_{k\in\mathbb{N}}|X-Y|>\frac{1}{k}\Big)\le\sum_{k\in\mathbb{N}}\mathbb{P}(|X-Y|>\frac{1}{k})=0.$

(ii) Note that in principle X and Y could even be defined on different probability spaces, and $X=Y$ a.s. would make no sense at all. But even taking the same probability space, the claim is false. As a counterexample take the probability space $([0,1],\mathbb{B}([0,1]),\lambda)$, and the random variables

$$X_n(x)=X(x)=x \quad\text{and}\quad Y(x)=x.$$

Obviously we have $X_n \xrightarrow{w} X$. Note that, for $d\in[0,1]$,

$$\lambda(X\le d)=\lambda([0,d])=d=\lambda([1-d,1])=\lambda(Y\le d),$$

so that $X\sim Y$ (X is equal to Y in distribution). But then $X_n\xrightarrow{w} Y$ too, even if $X\neq Y$ a.s..

Solution for Exercise 5.2

First of all the hint is true because of the following chain of equivalences:

$$\begin{aligned}\sum_{n=1}^{\infty}\mathbb{P}(|X_n|>n^{\frac{1}{\beta}})<\infty \quad&\iff\quad \sum_{n=1}^{\infty}\mathbb{P}(|X_1|>n^{\frac{1}{\beta}})<\infty\\ &\iff\quad \sum_{n=1}^{\infty}\mathbb{P}(|X_1|^{\beta}>n)<\infty\\ &\iff\quad \int_0^{\infty}\mathbb{P}(|X_1|^{\beta}>y)\,\mathrm{d}y<\infty\\ &\iff\quad \mathsf{E}[|X_1|^{\beta}]<\infty.\end{aligned}$$

In the first line we have used the i.i.d.-ness of the X_i's, while in the third line the fact that $\mathbb{P}(|X_1|^\beta > y)$ is monotone decreasing.
Now note that by Borel-Cantelli and the independence of the X_i's, the condition of the hint is true if and only if

$$\mathbb{P}(|X_n| > n^{1/\beta} \text{ for infintely many } n) = 0.$$

We prove now that, in fact, $\mathbb{P}(\lim_{n\to\infty} |X_n|/n^{1/\beta} = 0) = 1$ (this implies that the probability above is indeed 0):

$$\frac{X_n}{n^{1/\beta}} = \frac{S_n - S_{n-1}}{n^{1/\beta}} = \frac{S_n}{n^{1/\beta}} - \frac{(n-1)^{1/\beta}}{n^{1/\beta}} \frac{S_{n-1}}{(n-1)^{1/\beta}} \to 0 \quad \text{a.s.}$$

since $\frac{S_n}{n^{1/\beta}} \to 0$, $\frac{(n-1)^{1/\beta}}{n^{1/\beta}} \to 1$ and $\frac{S_{n-1}}{(n-1)^{1/\beta}} \to 0$ a.s..

Solution for Exercise 5.3
Assume that as $i \to \infty$, $E[X_i]$ does not converge to EX. Then there exists an $a \neq EX$ and a subsequence $(X_{n_k})_{k=1}^\infty$ such that $|a| \leq EY$ and

$$\lim_{k\to\infty} E[X_{n_k}] = a. \tag{*}$$

However, since $(X_{n_k})_{k=1}^\infty$ converges in probability to X, it has a subsequence $(X_{m_k})_{k=1}^\infty$ that converges almost surely to X (Theorem 2.14 in the script). Noting that $|X_{m_k}| < Y$ and by the dominated convergence theorem, we have

$$\lim_{k\to\infty} E[X_{m_k}] = E[X],$$

which contradicts with (*). Our proof is complete.

Solution for Exercise 5.4

(i) $\subseteq$: If the $\overline{\lim}$ of X_n is strictly smaller than c, then it stays under that value from some $N \in \mathbb{N}$ on. This implies that $\{X_n < c\}$ happens infinitely often. The vice-versa is not true: it is enough to consider the sequence $X_n = (-1)^n$ and $c = 0$.

(ii) $\subseteq$: If the $\overline{\lim}\, X_n$ is strictly smaller than c, than there exists $\varepsilon > 0$ and $N \in \mathbb{N}$ such that, for all $n > N$, we have $X_n < c - \varepsilon$. In particular this implies that from that N on $X_n < c$, so that the $\underline{\lim}$ on the right hand side is also satisfied. On the other hand, the deterministic sequence $X_n = c - \frac{1}{n}$ is such that the first event is the empty set (in fact, $\overline{\lim}\, X_n = c$), while the second event is the whole of Ω (X_n is always strictly smaller than c).

(iii) $\supseteq$: If the $\underline{\lim}$ happens, then X_n is going to be smaller of equal to c from some $N \in \mathbb{N}$ on, and in particular the *limsup* of the sequence will have to be smaller or equal of c, too. On the other hand, consider $X_n = c + \frac{1}{n}$. Then $\overline{\lim}\, X_n = c$, but it is not true that from some point on X_n is smaller or equal to c.