Solutions for Exercise sheet 6

Convergence of random variables

Solution for Exercise 6.1

First of all note that $\log(X_1) \in [1, \log(2)]$ almost surely and hence $Var(\log(X_1)) < \infty$. $(\log(X_i))_{i\in\mathbb{N}}$ is a sequence of i.i.d. random variables.
In order to use the Strong Law of Large Numbers we turn the product in a sum:

$$\lim_{n\to\infty} \Big(\prod_{1\le i\le n} X_i \Big)^{1/n} = c \quad \text{a.s.} \iff \lim_{n\to\infty} \frac{1}{n} \sum_{1\le i\le n} \log(X_i) = \log c \quad \text{a.s.}.$$

The Strong Law of Large Numbers tells us now that almost surely

$$\frac{1}{n} \sum_{1\le i\le n} \log(X_i) \to \mathsf{E}[\log(X_1)].$$

We are only left to find the value of c:

$$\mathsf{E}[\log(X_1)] = \int_1^2 \log(x)\,\mathrm{d}x = [x\log(x) - x]_{x=1}^{x=2} = 2\log(2) - 1,$$

from which follows

$$c = \exp(2\log(2) - 1) = \frac{4}{\mathrm{e}}.$$

Solution for Exercise 6.2

We will use the condition for convergence in distribution given by the exercise in section 2.4.4 of Prof. Berger's script. For $z \in \mathbb{R}^+$ (and eventually n big enough)

$$\begin{aligned} F_{Z_n}(z) &= \mathbb{P}(Z_n < z) \\ &= \mathbb{P}(n(\alpha - Y_n) < z) \\ &= \mathbb{P}(Y_n > \alpha - \tfrac{z}{n}) \\ &= 1 - \mathbb{P}(X_i \le \alpha - \tfrac{z}{n} \text{ for all } i = 1, ..., n) \\ &= 1 - \mathbb{P}(X_1 \le \alpha - \tfrac{z}{n})^n \\ &= 1 - \big(\tfrac{1}{\alpha}(\alpha - \tfrac{z}{n})\big)^n \\ &\xrightarrow{n\to\infty} 1 - \mathrm{e}^{-z/\alpha} = F_Z(z). \end{aligned}$$

If the last line we can recognize a well known distribution function: Z is an exponential random variable of parameter α^{-1}.

Solution for Exercise 6.3

(i) Clearly $\mathsf{E}[\frac{S_n}{n}] = 0$. We want to show that $\frac{S_n}{n}$ does not converge almost surely to 0. Call $A_n := \{|X_n| = n\}$. Then we have $\sum_{n=2}^{\infty} \mathbb{P}(A_n) = \infty$. This, together with the independence of the X_n's allows us to use the second Borel-Cantelli Lemma, which ensures that A_n happens infinitely often almost surely. This fact implies

$$\mathbb{P}(\tfrac{S_n}{n} \to 0) = 0.$$

In fact, every time A_n happens, the respective X_n gives a contribute of $+1$ or -1 to the sequence $\frac{S_n}{n}$. By, e.g., the Cauchy criterium, the sequence $\frac{S_n}{n}$ almost surely does not converge to 0.

(ii) We want now to prove the weaker statement $\mathbb{P}(|\frac{S_n}{n}| > \varepsilon) \xrightarrow{n\to\infty} 0$.

By Markov inequality

$$\mathbb{P}(|\tfrac{S_n}{n}| > \varepsilon) \leq \frac{\mathsf{E}[|\frac{S_n}{n}|^2]}{\varepsilon^2} = \frac{Var(S_n)}{n^2\varepsilon^2}.$$

Thanks to the independence of the X_n's we can calculate

$$\begin{aligned} Var(S_n) &= \sum_{k=2}^{n+1} Var(X_n) \\ &= \sum_{k=2}^{n+1} k^2 \frac{1}{k \log k} \\ &\leq n \frac{(n+1)^2}{(n+1)\log(n+1)} \leq 2\frac{n^2}{\log n}, \end{aligned}$$

where for the last line each summand was estimated with the biggest of them.

Putting all together we see that

$$\mathbb{P}(|\tfrac{S_n}{n}| > \varepsilon) \leq 2\frac{n^2}{\log n} \cdot \frac{1}{n^2\varepsilon^2} \xrightarrow{n\to\infty} 0.$$

Solution for Exercise 6.4

Take the space L^2 of all the random variables with finite second moment. Then $S_n \in L^2$ for all $n \in \mathbb{N}$. Now we show that S_n is in fact a Cauchy sequence in this space:

$$\begin{aligned} \mathsf{E}[(S_{n+m} - S_n)^2] &= \mathsf{E}\Big[\Big(\sum_{i=n+1}^{n+m} X_i\Big)^2\Big] \\ &= Var\Big(\sum_{i=n+1}^{n+m} X_i\Big) \\ &= \sum_{i=n+1}^{n+m} Var(X_i) \\ &= \sum_{i=n+1}^{n+m} \mathsf{E}[X_i^2] \xrightarrow{n\to\infty} 0. \end{aligned}$$

Since L^2 is compete, there exist a random variable $S \in L^2$ such that $S_n \xrightarrow{L^2} S$. But this implies also $S_n \xrightarrow{\mathbb{P}} S$.