Solutions for Exercise sheet 7

Characteristic functions and CLT

Solution for Exercise 7.1

(i) Note that the density of X_n tends rapidly to 0 unless t is very close to 1. It is therefore natural to investigate the convergence of $(X_n)_{n\in\mathbb{N}}$ to 1. We verify that in fact $(X_n)_{n\in\mathbb{N}}$ tends to 1 in Probability. For any $\varepsilon > 0$ we have

$$\begin{aligned}\mathbb{P}(|X_n - 1| > \varepsilon) &= \mathbb{P}(X_n > \varepsilon + 1) + \mathbb{P}(X_n < 1 - \varepsilon) \\ &= \mathbb{P}(X_n < 1 - \varepsilon) \\ &= \int_0^{1-\varepsilon} n t^{n-1}\,\mathrm{d}t \\ &= (1-\varepsilon)^n \xrightarrow{n\to\infty} 0,\end{aligned}$$

where in the second line we used the fact that X_n has only values in $[0,1]$ (just look at the density!).

(ii) Convergence in Probability implies convergence in distribution, hence $(X_n)_{n\in\mathbb{N}}$ tends to the random variable constantly equal to 1 also in distribution.

(iii) It is probably harder to understand what the exercises asks than to come up with a solution after that. If $X_n(\omega) = \omega^{1/n}$, then clearly, for λ-a.e. ω, $X_n \to 1$ (this is in fact true for all $\omega \in (0,1]$). We are left to verify that such X_n's have indeed the right density function: For all $x \in [0,1]$

$$\mathbb{P}(X_n(\omega) \le x) = \mathbb{P}(\omega^{1/n} \le x) = \mathbb{P}(\omega \le x^n) = x^n = \int_0^x f_n(t)\,\mathrm{d}t.$$

Solution for Exercise 7.2

First of all note that $|\varphi(u)| = 1$ iff $\exists r \in \mathbb{R}$ such that $\varphi(u) = \exp(ir)$.
"$\Rightarrow$": We choose $u = \frac{2\pi}{b}$. We see that

$$\begin{aligned}\mathsf{E}[\mathrm{e}^{iuX}] &= \sum_{z\in\mathbb{Z}} \mathrm{e}^{iu(a+bz)}\mathbb{P}(X = a+bz) \\ &= \mathrm{e}^{iua}\sum_{z\in\mathbb{Z}} \mathrm{e}^{i2\pi z}\mathbb{P}(X = a+bz) \\ &= \mathrm{e}^{iua}\sum_{z\in\mathbb{Z}} \mathbb{P}(X = a+bz) \\ &= \mathrm{e}^{iua},\end{aligned}$$

and we are done ($r = ua$).
"$\Leftarrow$": Since $|\varphi(u)| = 1$, there exists $r \in \mathbb{R}$ such that $\varphi(u) = \exp(ir)$.

$$\mathsf{E}[e^{iuX}] = e^{ir} \quad \Longleftrightarrow \quad \mathsf{E}[e^{i(uX-r)}] = 1.$$

Let $X' := X - \frac{r}{u}$. Then we have $\varphi_{X'}(u) = 1$ and

$$\mathsf{E}[e^{iuX'}] = \mathsf{E}[\cos(uX')] + i\mathsf{E}[\sin(uX')] = 1.$$

It follows that $\cos(uX')$ has to be almost surely equal to 1, and hence $uX' \in 2\pi\mathbb{Z}$ almost surely. Going back to the original random variable X, we have obtained $X \in \frac{2\pi}{u}\mathbb{Z} + \frac{r}{u}$, which is what we had to prove.

Solution for Exercise 7.3

(i) Let $X \sim Bin(n,p)$. Then

$$\begin{aligned}\varphi_X(t) = \mathsf{E}[e^{itX}] &= \sum_{k=0}^{n} e^{itk}\mathbb{P}(X = k) \\ &= \sum_{k=0}^{n} \binom{n}{k} (pe^{it})^k (1-p)^{n-k} \\ &= (1 - p + pe^{it})^n,\end{aligned}$$

since the last but one line is just the binomial expansion of the last expression.
Now let $Y \sim Po(\lambda)$. We have

$$\begin{aligned}\varphi_Y(t) = \mathsf{E}[e^{itY}] &= \sum_{k=0}^{\infty} e^{itk} \frac{\lambda^k}{k!} e^{-\lambda} \\ &= e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda e^{it})^k}{k!} \\ &= e^{\lambda(e^{it}-1)}.\end{aligned}$$

(ii) From part (i) we have immediately that

$$\varphi_{Y_{n,k}}(t) = \Big(1 + \frac{1}{n2^k}(e^{it} - 1)\Big)^n.$$

By the independence, we have therefore that

$$\varphi_{Z_n}(t) = \prod_{k=1}^{n} \varphi_{Y_{n,k}}(t) = \prod_{k=1}^{n} \Big(1 + \frac{1}{n2^k}(e^{it} - 1)\Big)^n$$

and

$$\log(\varphi_{Z_n}(t)) = \sum_{k=1}^{n} n \log\Big(1 + \frac{1}{n2^k}(e^{it} - 1)\Big).$$

Call $\alpha_k(t) := \frac{1}{2^k}(\mathrm{e}^{it} - 1)$ and note that $\left|\frac{\alpha_k(t)}{n}\right| < \frac{1}{2}$ for n big enough (depending on t, which is anyway fixed). Since, for $z \leq \frac{1}{2}$, we have

$$|\log(1+z) - z| \leq |z|^2,$$

it follows that

$$|\log \varphi_{Z_n}(t) - \sum_{k=1}^n \alpha_k(t)| \leq \sum_{k=1}^n n\Big|\log\Big(1 + \frac{\alpha_k(t)}{n}\Big) - \frac{\alpha_k(t)}{n}\Big| \leq \sum_{k=1}^n \frac{|\alpha_k(t)|^2}{n} \leq \frac{|t|^2}{3n}$$

where for the last inequality we have used the fact that $\sum_{k=1}^n 2^{-2k} \leq \sum_{k=1}^\infty 2^{-2k} = \frac{1}{3}$.

Now, since

$$\lim_{n\to\infty} \sum_{k=1}^n \alpha_k(t) = e^{it} - 1,$$

we finally have

$$\begin{aligned} |\log \varphi_{Z_n}(t) - (\mathrm{e}^{it} - 1)| &\leq |\log \varphi_{Z_n}(t) - \sum_{k=1}^n \alpha_k(t)| + |\sum_{k=1}^n \alpha_k(t) - (\mathrm{e}^{it} - 1)| \\ &\leq \frac{|t|^2}{3n} + |\sum_{k=1}^n \alpha_k(t) - (\mathrm{e}^{it} - 1)| \xrightarrow{n\to\infty} 0. \end{aligned}$$

This shows that for every $t \in \mathbb{R}$ we have

$$\lim_{n\to\infty} \varphi_{Z_n}(t) = \mathrm{e}^{\mathrm{e}^{it}-1},$$

so that Z_n converges in distribution to a Poisson random variable of parameter 1 (see part (i)).

Solution for Exercise 7.4

Call X_i the random variable associated to φ_i. Let N be a random variable independent of the X_i's with values in $\mathbb{N}$ and $\mathbb{P}(N = n) = \lambda_n$. Assume that also the X_i's are independent of each other. The characteristic function of $X_N := \sum_{n\geq 0} 1_{\{n\}}(N) X_n$ is

$$\begin{aligned} \mathsf{E}[\mathrm{e}^{itX_N}] &= \mathsf{E}\big[\sum_{n\geq 0} 1_{N=n}\mathrm{e}^{itX_n}\big] \\ &= \sum_{n\geq 0} \lambda_n \mathsf{E}\big[\mathrm{e}^{itX_n}\big] \\ &= \sum_{n\geq 0} \lambda_n \varphi_n(t). \end{aligned}$$