Solutions for Exercise sheet 8

Conditional expectation and Martingales

Solution for Exercise 8.1

(i)

$$X + Y = \mathsf{E}[X + Y | X + Y] = \mathsf{E}[X|X + Y] + \mathsf{E}[Y|X + Y] = 2\mathsf{E}[X|X + Y],$$

because of symmetry. Rearranging we obtain

$$\mathsf{E}[X|X + Y] = \frac{X + Y}{2}.$$

(ii)

$$\mathbb{P}(Z = n) = \sum_{k=1}^{n} \mathbb{P}(X_1 = k)\mathbb{P}(X_2 = n - k) = \mathrm{e}^{-(\lambda_1+\lambda_2)} \frac{(\lambda_1 + \lambda_2)^n}{n!},$$

which means that the sum of two independent Poisson r.v.'s is distributed again as a Poisson r.v. with parameter the sum of the two parameters. Now, for $k \le n$, we have

$$\begin{aligned}\mathbb{P}(X = k|Z = n) &= \frac{\mathbb{P}(X_1 = k,\, X_2 = n - k)}{\mathbb{P}(Z = n)} \\ &= \binom{n}{k}\Big(\frac{\lambda_1}{\lambda_1 + \lambda_2}\Big)^k \Big(1 - \frac{\lambda_1}{\lambda_1 + \lambda_2}\Big)^{n-k}.\end{aligned}$$

We recognize that this is nothing but the distribution of a Binomail random variable of parameters n and $p = \lambda_1/(\lambda_1 + \lambda_2)$. Hence the expectation of X conditioned on $Z = n$ is $np = n\lambda_1/(\lambda_1 + \lambda_2)$. Form this we deduce that the random variable $\mathsf{E}[X|Z]$ has the value

$$\mathsf{E}[X|Z] = Z\frac{\lambda_1}{\lambda_1 + \lambda_2}.$$

Solution for Exercise 8.2

First of all note that $\mathsf{E}[X_i] = \frac{2}{\lambda}$. This can be calculated explicitly (use an integration by part for solving the integral) or by noticing that X_i is distributed as the sum of two independent Exponential random variables of parameter λ.
Letting $S_n := \sum_{k=1}^{n} X_k$, the Central Limit Theorem tells us that $Z_n := \frac{1}{\sqrt{\sigma^2 n}}(S_n - \frac{2n}{\lambda})$, where σ^2 is the variance of the X_i's, converges in distribution to a standard Normal random variable $\mathcal{N}(0, 1)$. Therefore

$$\mathbb{P}\big(X_1 + ... + X_n \ge \tfrac{2n}{\lambda}\big) = \mathbb{P}(Z_n \ge 0) \xrightarrow{n\to\infty} \frac{1}{2}.$$

Solution for Exercise 8.3

A smart strategy is to condition on X_{n-1}: If we had $n-1$ consecutive tails after X_{n-1} flips, then with probability p we will have n consecutive tails at the next flip, while with probability $1-p$ we will have to restart afresh. In formulas:

$$\mathsf{E}[X_n] = \mathsf{E}[\mathsf{E}[X_n|X_{n-1}]] = \mathsf{E}[p(X_{n-1}+1) + (1-p)(X_{n-1}+1+\hat{X}_n)],$$

where $\hat{X}_n$ has the same distribution of X_n.
Using the linearity of the expectation and reshuffling this expression, we obtain

$$\mathsf{E}[X_n] = \frac{1+\mathsf{E}[X_{n-1}]}{p}.$$

This is a recursive formula that has $\mathsf{E}[X_1] = p^{-1}$ as a first step. Solving it, we finally obtain

$$\mathsf{E}[X_n] = \frac{p^{n-1}+p^{n-2}+...+p+1}{p^n}.$$

Solution for Exercise 8.4

By Jensen's inequality, $E[X^2|\mathcal{G}] \geq Y^2$, a.s.. Thus $E[Y^2] \leq E[X^2] < \infty$. For any $\mathcal{G}$-measurable Z with $E[Z^2] < \infty$,

$$E[(X-Z)^2] = E[(X-Y)^2] + E[(Y-Z)^2] + 2E[(X-Y)(Y-Z)].$$

Since by the tower property $E[(X-Y)(Y-Z)] = E[E[(X-Y)(Y-Z)|\mathcal{G}]] = 0$, we have

$$E[(X-Z)^2] = E[(X-Y)^2] + E[(Y-Z)^2] \geq E[(X-Y)^2].$$

The equality holds if and only if $Y = Z$, almost surely.