Solutions for Exercise sheet 9

Conditional expectation and Martingales

Solution for Exercise 9.1

First of all we compute the probability density function of Y conditioned on the event $\{X = n\}$:

$$
\begin{aligned}
f_{Y|X=n}(y|X=n) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\mathbb{P}(X=n,\, Y \le t)}{\mathbb{P}(X=n)}\Big)\Big|_{t=y} \\
&= \frac{b}{\mathbb{P}(X=n)}\frac{(ay)^n}{n!}\exp\{-(a+b)y\}.
\end{aligned}
$$

Moreover

$$
\begin{aligned}
\mathbb{P}(X=n) = \mathbb{P}(X=n,\, Y<\infty) &= b\int_o^\infty \frac{(ay)^n}{n!}\exp\{-(a+b)y\}\,\mathrm{d}y \\
&= \frac{ba^n}{(a+b)^{n+1}}\int_o^\infty \frac{(a+b)^{n+1}}{n!}y^n\exp\{-(a+b)y\}\,\mathrm{d}y \\
&= \frac{ba^n}{(a+b)^{n+1}},
\end{aligned}
$$

where in the last line we have used the hint: We are in fact integrating the density function of a Gamma$(n+1,\, a+b)$. We have obtained

$$
f_{Y|X=n}(y|X=n) = \frac{b}{\frac{ba^n}{(a+b)^{n+1}}}\frac{(ay)^n}{n!}\exp\{-(a+b)y\} = \frac{(a+b)^{n+1}}{n!}y^n\exp\{-(a+b)y\}1_{\{y\in(o,+\infty\}}.
$$

In other words, the law of Y conditioned on the event $\{X=n\}$ is a Gamma$(n+1, a+b)$. If $h : (0,\infty) \to [0,\infty]$ is a Borel function, then $\mathsf{E}[h(Y)|X=n] = \mathsf{E}[h(G)]$ where G is another random variable, defined on the same probability space, with law Gamma$(n+1, a+b)$, i.e.

$$
\mathsf{E}[h(Y)|X=n] = \frac{(a+b)^{n+1}}{n!}\int_0^\infty h(y)y^n\exp\{-(a+b)y\}\,\mathrm{d}y.
$$

We now compute $\mathsf{E}[Y/(X+1)]$:

$$
\begin{aligned}
\mathsf{E}\Big[\frac{Y}{X+1}\Big] &= \sum_{n=0}^{\infty} \mathsf{E}\Big[\frac{Y}{X+1}\Big| X=n\Big]\mathbb{P}(X=n) \\
&= \sum_{n=0}^{\infty} \mathsf{E}\Big[\frac{Y}{n+1}\Big]\mathbb{P}(X=n) \\
&= \sum_{n=0}^{\infty} \Big(\frac{n+1}{a+b}\Big)\frac{1}{n+1}\mathbb{P}(X=n) \\
&= \frac{1}{a+b}.
\end{aligned}
$$

Solution for Exercise 9.2

Define recursively $V_1 = \mathsf{E}[X_1]$ and

$$V_{n+1} = \mathsf{E}[X_{n+1} - X_n + V_n|\mathcal{F}_n].$$

V_n is clearly $\mathcal{F}_n$-mesurable and we have

$$
\begin{aligned}
\mathsf{E}[X_{n+1} - V_{n+1}|\mathcal{F}_n] = \mathsf{E}[X_{n+1}|\mathcal{F}_n] - V_{n+1} &= \mathsf{E}[X_{n+1}|\mathcal{F}_n] - \mathsf{E}[X_{n+1} - X_n + V_n|\mathcal{F}_n] \\
= \mathsf{E}[X_{n+1}|\mathcal{F}_n] - \mathsf{E}[X_{n+1}|\mathcal{F}_n] + X_n - V_n &= X_n - V_n.
\end{aligned}
$$

Note that $(X_n - V_n)$ is $\mathcal{F}_n$-measurable and is integrable.

Solution for Exercise 9.3

This is known as Polya's urn and if you go on in science it will surely pop out again in your career.
Let G_n and R_n be the numbers of green and red balls at time n, respectively. Then $X_n = \frac{G_n}{2+n}$. By definition (recall that $P(A|\mathcal{F}) := E[1_A|\mathcal{F}]$),

$$P\Big(X_{n+1} = \frac{G_n+1}{3+n}\Big| X_n\Big) = X_n$$

and

$$P\Big(X_{n+1} = \frac{G_n}{3+n}\Big| X_n\Big) = 1 - X_n.$$

Hence

$$
\begin{aligned}
E\Big[X_{n+1}\Big|X_n\Big] &= \frac{G_n+1}{3+n}P\Big(X_{n+1} = \frac{G_n+1}{3+n}\Big|X_n\Big) + \frac{G_n}{3+n}P\Big(X_{n+1} = \frac{G_n}{3+n}\Big|X_n\Big) \\
&= \frac{G_n + X_n}{3+n} \\
&= X_n.
\end{aligned}
$$

After we have conviced ourselves of the measurability and integrability of (X_n) we conclude that it is a martingale.

Solution for Exercise 9.4

Following the hint, we will first of all show that $\mathbb{P}(T > kN) \leq (1-\varepsilon)^k$ for all $k \geq 0$. We have

$$\begin{aligned}\mathbb{P}(T > kN) &= \mathbb{P}(T > N, T > 2N, ..., T > kN) \\ &= \mathsf{E}[1_{T>N,...,T>kN}] \\ &= \mathsf{E}[1_{T>N} \cdot ... \cdot 1_{T>kN}] \\ &= \mathsf{E}[\mathsf{E}[1_{T>N} \cdot ... \cdot 1_{T>kN} | \mathcal{F}_N]] \\ &= \mathsf{E}[1_{T>N}\mathsf{E}[\cdot ... \cdot 1_{T>kN} | \mathcal{F}_N]],\end{aligned}$$

where we have used the measurability of $1_{T>N}$ w.r.t. $\mathcal{F}_N$, since T is a stopping time. Iterating this procedure conditioning on $\mathcal{F}_{2N}$, $\mathcal{F}_{3N}$... we obtain

$$\begin{aligned}\mathbb{P}(T > kN) &= \mathsf{E}[1_{T>N}\mathsf{E}[1_{T>2N}...\mathsf{E}[1_{T>kN}|\mathcal{F}_{(k-1)N}]...|\mathcal{F}_N]] \\ &\leq \mathsf{E}[1_{T>N}\mathsf{E}[1_{T>2N}...(1-\varepsilon)...|\mathcal{F}_N]] \\ &\leq (1-\varepsilon)^k.\end{aligned}$$

By a well known formula for the expectation of positive random variables (Ex. 2.2 (ii))

$$\mathsf{E}[T] = \sum_{n=0}^{\infty} \mathbb{P}(T \geq n) \leq \sum_{n=0}^{\infty} \mathbb{P}(T \geq k_n) = N \sum_{k=0}^{\infty} \mathbb{P}(T \geq kN) \leq N \sum_{k=0}^{\infty} (1-\varepsilon)^k = \tfrac{N}{\varepsilon} < \infty,$$

where we have used the notation $k_n = \lfloor n/N \rfloor$.