TU München
Zentrum Mathematik
Lehrstuhl für Wahrscheinlichkeitstheorie

Stochastic Analysis
Winter Term 2011/12
Gantert/Kochler

Exercise sheet 0

You do not need to hand in your solutions. We will work on this sheet together in the first tutorial.

Exercise 0.1

Consider a sequence $(X_n)_{n\in\mathbb{N}}$ of i.i.d. random variables with $\mathsf{E}[X_1] = 1$, $X_1 \geq 0$ and $\mathsf{P}(X_1 = 1) < 1$.

(i) Show that the sequence $(M_n)_{n\in\mathbb{N}_0}$ with

$$M_0 := 1$$
$$M_n := \prod_{i=1}^{n} X_i \quad \text{for } n \geq 1$$

is a martingale with respect to the filtration $\big(\mathcal{A}_n\big)_{n\in\mathbb{N}_0} := \big(\sigma(X_i : \ 1 \leq i \leq n)\big)_{n\in\mathbb{N}_0}$.

(ii) Show that $(M_n)_{n\in\mathbb{N}_0}$ converges a.s. for $n \to \infty$ and determine its limit.

(iii) Is $(M_n)_{n\in\mathbb{N}_0}$ uniformly integrable?

Exercise 0.2

Let us consider the following model which is known as Polya's urn:
At time 0 we start with r red and b black ball in the urn. Afterwards, we draw one ball with uniform probability out of the urn at every time $n \in \mathbb{N}$ which we then return along with c new ball of the same colour to the urn.

It will be helpful to consider the sequence $(X_n)_{n\in\mathbb{N}}$ with

$$X_n := \begin{cases} 1 & \text{if we draw a red ball at time } n \\ 0 & \text{otherwise.} \end{cases}$$

(i) Let us denote the proportion of red balls in the urn after the n-th drawing by M_n. Show that $(M_n)_{n\in\mathbb{N}_0}$ is a martingale with respect to the filtration

$$\big(\mathcal{A}_n\big)_{n\in\mathbb{N}_0} := \big(\sigma(X_i : \ 1 \leq i \leq n)\big)_{n\in\mathbb{N}_0}.$$

(ii) Does $(M_n)_{n\in\mathbb{N}_0}$ converge a.s. for $n \to \infty$?

Please turn the page!

Exercise 0.3

Consider the following basic model of a branching process in discrete time:
Let $(Y_{k,n})_{k,n\in\mathbb{N}}$ be a set of i.i.d. random variables with values in $\mathbb{N}_0$ and $0 < m < \infty$ for $m := \mathsf{E}\big[Y_{1,1}\big]$. Now we can consider the branching process $(Z_n)_{n\in\mathbb{N}_0}$ where

$$Z_0 := 1$$

$$Z_{n+1} := \sum_{k=1}^{Z_n} Y_{k,n+1} \quad \text{for } n \geq 0.$$

(i) Show that

$$\left(\frac{Z_n}{m^n}\right)_{n\in\mathbb{N}_0}$$

is a martingale with respect to the filtration $(\mathcal{A}_n)_{n\in\mathbb{N}_0}$ where

$$\mathcal{A}_n := \sigma(Y_{k,\ell} : \ k \in \mathbb{N}, \ 1 \leq \ell \leq n).$$

(ii) Show that $\left(\frac{Z_n}{m^n}\right)_{n\in\mathbb{N}_0}$ converges for $n \to \infty$.

(iii) What can you say about $\lim\limits_{n\to\infty} Z_n$ because of (ii)?