TU München
Zentrum Mathematik
Lehrstuhl für Wahrscheinlichkeitstheorie

Stochastic Analysis
Winter Term 2011/12
Gantert/Kochler

Exercise sheet 1

Exercises 1.3 and 1.4 are due in the lecture on October 27.

Please use the following two exercises to refresh your knowledge of conditional expectations and martingales. You do not need to hand in your solutions:

Exercise 1.1

Please repeat what you have learned about conditional expectations and martingales! For example, you can read chapter 13 and chapter 14 of the notes on probability theory for which you can find a link online:

http://www-m14.ma.tum.de/en/teaching/stochastic-analysis/

Exercise 1.2

Please answer the following questions without looking in your notes:

(i) Write down the two characterizing properties of the conditional expectation.

(ii) Consider two random variables X, Y with $\mathsf{E}\big[|X|^2\big] < \infty$ and assume that X and Y are independent. Compute $\mathsf{E}\big[X^2\big|Y\big]$ and $\mathsf{E}\big[X^2\big|X\big]$.

(iii) Consider two random variables X, Y with $\mathsf{E}\big[|X^4Y^7|\big] < \infty$. Compute $\mathsf{E}\big[X^4Y^7\big|Y\big]$.

(iv) Consider a probability space $(\Omega, \mathcal{A}, \mathsf{P})$, a random variable $X \in \mathcal{L}^1(\mathsf{P})$ and two σ-algebras $\mathcal{A}_1, \mathcal{A}_2$ with $\mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \mathcal{A}$. Compute

$$\mathsf{E}\big[\mathsf{E}\big[X\big|\mathcal{A}_1\big]\big|\mathcal{A}_2\big] \quad \text{and}$$
$$\mathsf{E}\big[\mathsf{E}\big[X\big|\mathcal{A}_2\big]\big|\mathcal{A}_1\big].$$

(v) Consider a random variable $X \in \mathcal{L}^2(\mathsf{P})$ and show that we a.s. have

$$\mathsf{E}\big[X^2\big|\mathcal{A}\big] \geq \big(\mathsf{E}\big[X\big|\mathcal{A}\big]\big)^2.$$

(vi) Give the definition of a martingale $(M_n)_{n\in\mathbb{N}_0}$.

(vii) What do you know about $\mathsf{E}\big[M_{1073}\big]$ when $(M_n)_{n\in\mathbb{N}_0}$ is a martingale?

Please turn the page!

Exercise 1.3 (4 Points)
Consider two random variables X and Y which are i.i.d. with distribution $\mathcal{N}(0, \sigma^2)$ for some $\sigma^2 > 0$. Show that $X + Y$ and $X - Y$ are also i.i.d. with distribution $\mathcal{N}(0, 2\sigma^2)$.

Exercise 1.4 (6 Points)
Consider a sequence $(X_n)_{n\in\mathbb{N}}$ of i.i.d. random variables with

$$\mathsf{P}(X_1 = 1) = \mathsf{P}(X_1 = -1) = \frac{1}{2}.$$

We can now introduce the associated random walk $(S_n)_{n\in\mathbb{N}_0}$ where

$$S_0 := 0 \qquad \text{and}$$
$$S_n := \sum_{i=1}^{n} X_i \qquad \text{for } n \geq 1.$$

(i) Determine $c(n) \in \mathbb{R}$ for all $n \in \mathbb{N}_0$ such that

$$\big(S_n^2 - c(n)\big)_{n\in\mathbb{N}_0}$$

is a martingale with respect to the filtration $\big(\mathcal{A}_n\big)_{n\in\mathbb{N}_0} := \big(\sigma(X_i :\ 1 \leq i \leq n)\big)_{n\in\mathbb{N}_0}$.

(ii) For $d \in \mathbb{N}$ consider

$$T_d := \inf\{n \in \mathbb{N} :\ |S_n| = d\}.$$

Compute $\mathsf{E}\big[T_d\big]$.

Registration: You will be able to sign up for the exercise classes via TUMonline starting from Thursday, October 20, 18:00. We will only work on Exercise sheet 0 in the first two tutorials (Thursday, October 27, 8:15 - 9:45 and 16:15 - 17:45). Participants of the tutorial Group B are invited to join the first tutorial of Group A (Thursday, October 27, 8:15 - 9:45) since only the room for this tutorial is large enough.

Exercise sheets: Exercise sheets will be uploaded on Thursdays. Usually you will have two weeks to work on the exercise sheet. This first exercise sheet is an exception and the solutions are due next week. Please try to hand in your solutions in a group of two.

Bonus system: Your homework will be corrected and rated with points. Through the continuous participation in the exercise course you can receive a bonus on your exam grade. If you achieve at least 60 % of all achievable points, you will get a bonus of one grade level on the grade of your exam if passed (i.e., 1.7 becomes 1.3, 2.3 becomes 2.0, 3.0 becomes 2.7, etc.). Improving the grade 1.0 or failed exams is not possible. This bonus counts only for the two exams corresponding to this lecture.