Exercise sheet 2

The Exercises are due in the lecture on November 10.

Exercise 2.1 (5 Points)

(i) Consider a random vector

$$\vec{X} := \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix}$$

with a $\mathcal{N}(\vec{\mu}, \Sigma)$ distribution, some matrix $A \in \mathbb{R}^{k\times n}$ and some vector $\vec{b} \in \mathbb{R}^k$. Determine the distribution of $A\vec{X} + \vec{b}$.

(ii) Consider a two-dimensional random vector (X_1, X_2) with common density

$$f(x_1, x_2) := \frac{\sqrt{2}}{\pi} \exp\left(-\frac{3}{2}x_1^2 - x_1x_2 - \frac{3}{2}x_2^2\right).$$

Determine the marginal distributions of X_1 and X_2. Are X_1 and X_2 stochastically independent?

Exercise 2.2 (5 Points)
Consider a random variable X_1 with a $\mathcal{N}(0,1)$ distribution. Let Y be another random variable which is independent of X_1 and for which we have $\mathsf{P}(Y = 1) = \frac{1}{2} = \mathsf{P}(Y = -1)$. Further we define $X_2 := Y \cdot X_1$. Show that the following statements hold:

(i) X_2 has a $\mathcal{N}(0,1)$ distribution.

(ii) X_1 and X_2 are uncorrelated but not independent.

(iii) (X_1, X_2) does not have a two-dimensional normal distribution.

Exercise 2.3 (5 Points)
Let $(B_t)_{t\in[0,1]}$ be a Brownian motion on some probability space $(\Omega, \mathcal{A}, \mathsf{P})$. Show that the process $(X_t)_{t\in[0,1]}$ where

$$X_t := B_{1-t} - B_1 \quad \text{for } 0 \le t \le 1$$

is also a Brownian motion on the same probability space $(\Omega, \mathcal{A}, \mathsf{P})$.

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Exercise 2.4 (5 + 8 Points)
Consider the space $S := C[0,1]$ of continuous functions on $[0,1]$ equipped with the norm $\|x\| := \sup_{t\in[0,1]} |x(t)|$ (for $x \in S$) and let $\mathcal{B}_S$ denote the associated Borel σ-algebra on S.

(i) For $t \in [0,1]$ consider the function $g_t : S \to \mathbb{R}$ where $g_t(x) := x(t)$ (with $x \in S$). Let $\mathcal{F} := \sigma(\{g_t : \ 0 \leq t \leq 1\})$ and show that $\mathcal{F} = \mathcal{B}_S$.

(ii*) Let $C_b(S)$ denote the space of continuous and bounded functions on S. Show that $\mathcal{F} = \sigma(\{f : \ f \in C_b(S)\})$.

(iii*) Cylinder sets on S are sets of the form

$$\{x \in S : \ (x(t_1), x(t_2), \ldots, x(t_n)) \in A\} \text{ for } A \in \mathcal{B}^n, \ 0 \leq t_1 < t_2 < \ldots < t_n \leq 1$$

for some $n \in \mathbb{N}$. (Here $\mathcal{B}^n$ denotes the Borel σ-algebra on $\mathbb{R}^n$). Let $\mathcal{Z}$ denote the set of all cylinder sets on S and show that we have $\mathcal{F} = \sigma(\mathcal{Z})$.

Hint: You may use that we have

$$\mathcal{B}_S \overset{def}{=} \sigma\Big(A \subseteq S : A \text{ open in } \big(S, \|\cdot\|\big) \Big) = \sigma\Big(B(x,\varepsilon) : \ x \in S, \ \varepsilon > 0 \Big)$$

due to the Weierstrass approximation theorem. Here $B(x,\varepsilon)$ denotes the open ball around x with radius ε

$$B(x,\varepsilon) := \{y \in S : \ \|x - y\| < \varepsilon\}.$$

Further use that $\mathbb{Q} \cap [0,1]$ is dense in $[0,1]$.