Exercise sheet 4

The Exercises are due in the lecture on **December 1!**

Exercise 4.1 (5 + 2 Points)
Let $(B_t)_{t\in[0,1]}$ be a Brownian motion with continuous paths (i.e. $t \mapsto B_t(\omega)$ is continuous for *every* ω) and consider

$$L_t := \int_0^t B_s \, ds$$

for $0 \le t \le 1$.

(i) Show that L_t is a well defined random variable.

(ii) Compute $\mathsf{E}\big[L_t\big]$ and $\mathsf{Cov}(L_s, L_t)$ for $s, t \in [0,1]$.

(iii*) Show that $(L_t)_{t\in[0,1]}$ is a Gaussian process.

Exercise 4.2 (5 + 4 Points)
Consider a Brownian motion $B = (B_t)_{t\in[0,1]}$, a function $h \in C[0,1]$ with $h(0) = 0$ and a filtration $(\mathcal{A}_n)_{n\in\mathbb{N}}$ where

$$\mathcal{A}_n := \sigma(\{B_t : t \in \{j2^{-n} : j = 0, 1, \ldots, 2^n\}\}).$$

Further define

$$H_n := 2^{n-1} \sum_{j=1}^{2^n} \left((\Delta_j^{(n)} h)^2 - 2 \cdot (\Delta_j^{(n)} B) \cdot (\Delta_j^{(n)} h) \right)$$

where $\Delta_j^{(n)} x = x\left(\frac{j}{2^n}\right) - x\left(\frac{j-1}{2^n}\right)$ for $x \in C[0,1]$.

(i) Show that e^{-H_n} is a martingale with respect to the filtration $(\mathcal{A}_n)_{n\in\mathbb{N}}$.

Hint: Use the Brownian motion which we have constructed in the lecture. There we have

$$B_{\frac{2j-1}{2^n}} = \frac{1}{2} \cdot \left(B_{\frac{2j-2}{2^n}} + B_{\frac{2j}{2^n}} \right) + 2^{-\frac{n+1}{2}} \cdot Z_{\frac{2j-1}{2^n}}$$

where $(Z_t)_{t\in\mathcal{D}}$ are i.i.d. random variables with law $\mathcal{N}(0,1)$ for $\mathcal{D} = \bigcup_{n=0}^{\infty} \left\{ \frac{k}{2^n} : k = 0, 1, \ldots, 2^n \right\}$.
Try to express $\Delta_{2j-1}^{(n)} B$ and $\Delta_{2j}^{(n)} B$ with the help of $\Delta_j^{(n-1)} B$ and $Z_{\frac{2j-1}{2^n}}$.

Please turn the page!

From now on assume that the Brownian motion $B = (B_t)_{t\in[0,1]}$ has continuous paths:

(ii*) Consider $h \in \mathcal{H}$ where

$$\mathcal{H} := \left\{ h \in C[0,1] : \text{ there is } f \in L^2[0,1] \text{ such that } h(t) = \int\limits_0^t f(s)ds,\ 0 \le t \le 1 \right\}$$

and the associated distribution μ_h of $(B_t + h(t))_{t\in[0,1]}$ on $(C[0,1], \mathcal{F})$. Show that the Radon-Nikodym derivative $\frac{d\mu_h}{d\mu}$ is given by

$$\frac{d\mu_h}{d\mu} = \exp\left(-\frac{1}{2}\int_0^1 h'(s)^2 ds + \int_0^1 h' dB\right)$$

where μ denotes the distribution of the Brownian motion $(B_t)_{t\in[0,1]}$ on $(C[0,1], \mathcal{F})$.

Hint: Use the Radon-Nikodym derivatives $(\frac{d\mu_{h,n}}{d\mu_n})_n$ from the lecture. You may further use that $\big(\exp(-H_n)\big)_{n\in\mathbb{N}}$ is uniformly integrable for $h \in \mathcal{H}$ and therefore we a.s. have

$$\mathsf{E}\left[\lim_{k\to\infty} e^{-H_k} \,\middle|\, \mathcal{A}_n\right] = e^{-H_n}.$$

(iii) Simplify

$$\frac{d\mu_h}{d\mu}$$

if $h(t) = ct$ for some $c \in \mathbb{R}$.