Exercise sheet 5

The Exercises are due in the lecture on December 15.

Exercise 5.1 (5 Points)
Consider some measurable space $(\Omega, \mathcal{F})$ with a filtration $(\mathcal{F}_t)_{t\geq 0}$. For a stopping time T we have defined

$$\mathcal{F}_T := \{A \in \mathcal{F} : \ A \cap \{T \leq t\} \in \mathcal{F}_t \ \forall t \geq 0\}.$$

(a) (i) Show that $\mathcal{F}_T$ is a σ-field.
 (ii) Show that $\mathcal{F}_S \subseteq \mathcal{F}_T$ holds for two stopping times S, T with $S \leq T$.

(b) Show that $S \wedge T = \min\{S, T\}$ and $S \vee T = \max\{S, T\}$ are also stopping times if S and T are stopping times.

Exercise 5.2 (6 + 2 Points)
Consider a Brownian motion $(B_t)_{t\geq 0}$, the stopping time

$$T_c := \inf\{t \geq 0 : \ B_t = c\}$$

for $c \in \mathbb{R}$ and the martingale $(M_t^{(\alpha)})_{t\geq 0}$ with

$$M_t^{(\alpha)} := \exp\left(\alpha B_t - \frac{1}{2}\alpha^2 t\right)$$

for $\alpha \in \mathbb{R}$.

(i) Show that $\left(M_{T_c \wedge n}^{(\alpha)}\right)_{n \in \mathbb{N}_0}$ is a.s. bounded for $c, \alpha > 0$.

(ii) Show that $\mathsf{E}\big[\mathbb{1}_{\{T_c < \infty\}} \cdot M_{T_c}^{(\alpha)}\big] = 1$ for $c, \alpha > 0$.

(iii) Conclude that $\mathsf{P}(T_c < \infty) = 1$ for all $c \in \mathbb{R}$.

(iv*) Use (iii) to show that we a.s. have

$$\overline{\lim_{t\to\infty}} B_t = +\infty \quad \text{and} \ \underline{\lim_{t\to\infty}} B_t = -\infty.$$

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Exercise 5.3 (4 + 4 Points)
Consider a Brownian motion $(B_t)_{t\geq 0}$ and the random time

$$T_{a,b} := \inf\{t \geq 0 : \; B_t \in \{a, b\}\}$$

for $a < 0 < b$.

(i) Show that

$$\mathsf{P}\left(B_{T_{a,b}} = a\right) = \frac{b}{|a| + b} \quad \text{and} \quad \mathsf{P}\left(B_{T_{a,b}} = b\right) = \frac{|a|}{|a| + b}.$$

(ii*) Show that

$$\mathsf{E}\big[T_{a,b}\big] = |a| \cdot b.$$

Hint for (ii*): Recall Exercise 1.2.

Exercise 5.4 (5 Points)
Let $(B_t)_{t\geq 0}$ be a Brownian motion on some probability space $(\Omega, \mathcal{F}, \mathsf{P})$. Which of the following processes $(X_t)_{t\geq 0}$ is (and is not) a martingale, a submartingale or a supermartingale with respect to its own filtration $(\mathcal{F}_t)_{t\geq 0} = \sigma(\{X_s : \; 0 \leq s \leq t\})$?

(i) $X_t := (B_t)^4$ for $t \geq 0$.

(ii) $X_t := \exp(B_t - t)$ for $t \geq 0$.

(iii) $X_t := |B_t| - t$ for $t \geq 0$.