Exercise sheet 6

The Exercises are due in the lecture on January 12.

Exercise 6.1 (5 + 2 Points)
Consider a Brownian motion $(B_t)_{t\geq 0}$ and the stopping time

$$T_c := \inf\{t > 0 :\ B_t = c\}.$$

(i) Show that we have

$$\mathsf{E}[\exp(-\lambda T_c)] = \exp(-c\sqrt{2\lambda})$$

for $c, \lambda > 0$.

(ii) Conclude that $\mathsf{E}[T_c] = \infty$ for all $c \neq 0$.

(iii*) Compute $\mathsf{E}[T_0]$.

Hint for (i): Use the martingale from Exercise 3.4 (iii).
Hint for (iii*): Use what you have learned about the time inversion of the Brownian motion. Further use that every Brownian motion a.s. returns to 0 infinitely often according to Exercise 5.2 (iv).

Exercise 6.2 (5 Points)

(i) Consider a martingale $\big(M_n\big)_{n\in\mathbb{N}_0}$ with respect to some filtration $\big(\mathcal{A}_n\big)_{n\in\mathbb{N}_0}$ and a sequence of predictable random variables $\big(H_n\big)_{n\in\mathbb{N}}$, i.e. H_n is measurable with respect to $\mathcal{A}_{n-1}$ for every $n \in \mathbb{N}$. Let us define $\big((H.M)_n\big)_{n\in\mathbb{N}_0}$ by

$$(H.M)_0 := 0,$$
$$(H.M)_n := \sum_{j=1}^{n} H_j \cdot (M_j - M_{j-1}) \quad \text{for } n \geq 1.$$

Show that $\big((H.M)_n\big)_{n\in\mathbb{N}_0}$ is also a martingale with respect to the filtration $\big(\mathcal{A}_n\big)_{n\in\mathbb{N}_0}$ if we assume that

$$H_n \cdot (M_n - M_{n-1}) \in \mathcal{L}^1 \quad \text{for all } n \geq 1.$$

Remark: $\big((H.M)_n\big)_{n\in\mathbb{N}_0}$ is a discrete-time analogue of the stochastic integral.

(ii) $\big((H.M)_n\big)_{n\in\mathbb{N}_0}$ has a nice interpretation in terms of a game:
We can understand $M_n - M_{n-1}$ as the result of the n-th round in a game, H_n as the bet on the n-th round which has to be chosen before the n-th round and $(H.M)_n$ as the balance after the n-th round. For example, let us assume that we toss a fair coin at every time $n \in \mathbb{N}$. Before every round we have the chance to bet an arbitrary amount of money on the outcome of the next round. If the coin shows head then our bet will be doubled, otherwise we lose our bet.

Please turn the page!

(a) Assume that we bet $H_1 = 1$ in the first round. Construct a predictable sequence of bets $\left(H_n\right)_{n\in\mathbb{N}_0}$ such that we a.s. have

$$T := \sup\{n \in \mathbb{N} : \; H_n \neq 0\} < \infty \quad \text{and} \quad (H.M)_T = 1.$$

(b) Why is the result of (a) no contradiction to the optional stopping theorem?

Exercise 6.3 (5 Points)

(i) Consider a function $h \in \mathcal{H}$ and a Brownian motion $(B_t)_{t\in[0,1]}$. Show that the Paley-Wiener stochastic integral

$$\int_0^1 h' \, dB$$

defined in Lemma 3.4 almost surely agrees with the stochastic integral defined in chapter 5 of the lecture.

(ii) Determine the distribution of

$$\int_0^1 h' \, dB.$$

Exercise 6.4 (5 Points)

Suppose $H = \{H_s(\omega) : \; s \geq 0, \; \omega \in \Omega\}$ is progressively measurable and $(B_t)_{t\geq 0}$ a Brownian motion. Show that for any stopping time T with

$$\mathsf{E}\left[\int_0^T (H_s)^2 ds\right] < \infty$$

we have

(i) $\mathsf{E}\left[\int_0^T H_s dB_s\right] = 0,$

(ii) $\mathsf{E}\left[\left(\int_0^T H_s dB_s\right)^2\right] = \mathsf{E}\left[\int_0^T (H_s)^2 ds\right].$