TU München
Zentrum Mathematik
Lehrstuhl für Wahrscheinlichkeitstheorie

Stochastic Analysis
Winter Term 2011/12
Gantert/Kochler

Exercise sheet 7

The Exercises are due in the lecture on January 26.

Exercise 7.1 (6 Points)
Let $\left(B_t\right)_{t\geq 0}$ be a Brownian motion and consider the following stochastic integrals:

(i) $\int\limits_0^t s \, dB_s$

(ii) $\int\limits_0^t B_s \cdot e^{B_s} \, dB_s$

(iii) $\int\limits_0^t \frac{1}{1+(B_s)^2} \, dB_s$

Write them as functions of the Brownian motion $\left(B_t\right)_{t\geq 0}$ without a stochastic integral.

Exercise 7.2 (7 Points)
Let $\left(B_t\right)_{t\geq 0}$ be a Brownian motion. Find an increasing process $(A_t)_{t\geq 0}$ and functions $f : \mathbb{R} \to \mathbb{R}$ such that the following processes $\left(M_t\right)_{t\geq 0}$ are continuous (but not constant) martingales with respect to their own filtration $\left(\mathcal{F}_t\right)_{t\geq 0} := \sigma\left(\left\{M_s : \ 0 \leq s \leq t\right\}\right)_{t\geq 0}$:

(i) $M_t := (B_t)^4 - A_t \quad$ for $t \geq 0$

(ii) $M_t := f(t) \cdot \cos(B_t) \quad$ for $t \geq 0$

(iii) $M_t := (B_t + f(t)) \cdot \exp\left(-B_t - \frac{1}{2} \cdot t\right) \quad$ for $t \geq 0$

Exercise 7.3 (7 Points)
Consider the Hermite polynomials $\left(h_n(x,t)\right)_{n=0,1,\ldots}$ which are determined by the power series

$$\exp\left(\alpha x - \tfrac{1}{2}\alpha^2 t\right) = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} h_n(x,t) \quad \text{for } x \in \mathbb{R}, t \geq 0,$$

i.e.

$$h_n(x,t) = \frac{\partial^n}{\partial \alpha^n} \exp\left(\alpha x - \tfrac{1}{2}\alpha^2 t\right)\Big|_{\alpha=0}.$$

Please turn the page!

(i) Show that the Hermite polynomials $\big(h_n(x,t)\big)_{n=0,1,\dots}$ fulfill the differential equation

$$\frac{1}{2}\frac{\partial^2}{\partial x^2}h_n + \frac{\partial}{\partial t}h_n = 0$$

for every $n \in \mathbb{N}_0$.

(ii) Show that $M^{(n)} := \big(h_n(B_t,t)\big)_{t\geq 0}$ is a martingale for every $n = 0,1,\dots$ where $\big(B_t\big)_{t\geq 0}$ denotes a Brownian motion.

(iii) Compute the quadratic variation $\langle M^{(n)} \rangle_t$ in $t \geq 0$ for all $n \in \mathbb{N}$.

Hint for (iii): Show that $\frac{\partial}{\partial x}h_n = n \cdot h_{n-1}$ for $n \geq 1$.