TU München
Zentrum Mathematik
Lehrstuhl für Wahrscheinlichkeitstheorie

Stochastic Analysis
Winter Term 2011/12
Gantert/Kochler

Exercise sheet 8

You do not need to hand in your solutions. We will discuss the solutions in the tutorials on February 9.

Exercise 8.1
Let $(B_t)_{t\geq 0}$ be a Brownian motion. Find stochastic differential equations which are solved by the following processes $(Z_t)_{t\geq 0}$:

(i) $Z_t := \exp(B_t)$

(ii) $Z_t := \dfrac{B_t}{1+t}$

(iii) $Z_t := \exp\left(\alpha B_t + \left(\beta - \frac{\alpha^2}{2}\right) t\right) \quad$ for $t \geq 0$, where $\alpha, \beta \in \mathbb{R}$

Exercise 8.2
Let $(B_t)_{t\geq 0}$ be a Brownian motion and consider the processes $M := (M_t)_{t\geq 0}$ and $Z := (Z_t)_{t\geq 0}$ where

$$M_t := \exp\left(\alpha B_t - \tfrac{1}{2}\alpha^2 t\right)$$
$$Z_t := \int_0^t \frac{1}{M_s} dB_s$$

for $t \geq 0$ with some $\alpha \in \mathbb{R}$.

(i) Compute $E\left[\langle M\rangle_t\right]$ for $t \geq 0$.

(ii) Compute $\langle Z, M\rangle_t$ for $t \geq 0$.

Multiple choice

- For each correct cross you will receive 1 point and for each incorrect cross you will lose 1 point.
- If you do not check any cross for an assertion, then you will neither receive 1 point nor lose 1 point.
- You will receive at least 0 points for Exercise 8.3.

Please turn the page!

Exercise 8.3
Let $(B_t)_{t\geq 0}$ be a Brownian motion on some complete probability space $(\Omega, \mathcal{A}, \mathsf{P})$. Decide which of the following statements are true and mark them with a cross:

The random variables $B_{t_2} - B_{t_1}, B_{t_4} - B_{t_3}, \ldots, B_{t_{2n}} - B_{t_{2n-1}}$ with $t_1, t_2, \ldots, t_{2n} \geq 0$ are independent if and only if they are uncorrelated ...

☐ ... is true.

☐ ... is false.

$\left(B_t^2 - t\right)_{t\geq 0}$ is a Gaussian process ...

☐ ... is true.

☐ ... is false.

$\left(B_t^4 + t\right)_{t\geq 0}$ is a ...

☐ martingale.

☐ submartingale.

☐ supermartingale.

Let $A := \left\{\omega \in \Omega : \ \exists t \in [0,1] \text{ such that } B_t(\omega) = \sup\{B_s(\omega) : \ 0 \leq s \leq 1\}\right\}$. Then:

☐ $\mathsf{P}(A) = 0$

☐ $0 < \mathsf{P}(A) < 1$

☐ $\mathsf{P}(A) = 1$

Bonus question (only on this exercise sheet):

☐ $\exists s \in [0,1] : \ \mathsf{P}(B_s \geq B_t \ \forall t \in [0,1]) > 0$

☐ $\nexists s \in [0,1] : \ \mathsf{P}(B_s \geq B_t \ \forall t \in [0,1]) > 0$