Stochastic Analysis
Exercise Sheet 1

November 3/November 10, 2011

Fact 1:

Theorem

For a random variable $X \sim \mathcal{N}(\mu, \sigma^2)$, $a \in \mathbb{R}$ *and* $b \in \mathbb{R}$ *we have*

$$aX + b \sim \mathcal{N}(a\mu + b, a^2\sigma^2).$$

Remark

Here we allow the degenerate case of a constant and we say $c \sim \mathcal{N}(c, 0)$ *for every* $c \in \mathbb{R}$.

Fact 2:

Theorem

For two independent random variables $X \sim \mathcal{N}(\mu, \sigma^2)$ *and* $Y \sim \mathcal{N}(\nu, \rho^2)$ *we have*

$$X + Y \sim \mathcal{N}(\mu + \nu, \sigma^2 + \nu^2).$$

Fact 3:

Definition

A random vector $\vec{X} := (X_1, X_2, \ldots, X_n)^{tr}$ *has a (multivariate) normal distribution iff*

$$\langle (X_1, X_2, \ldots, X_n), (a_1, a_2, \ldots, a_n) \rangle = \sum_{i=1}^{n} a_i X_i$$

has a one-dimensional normal distribution for all $a_1, a_2, \ldots, a_n \in \mathbb{R}$*. We write* $\vec{X} \sim \mathcal{N}(\vec{\mu}, \Sigma)$ *where*

$$\vec{\mu} := (\mathrm{E}[X_1], \mathrm{E}[X_2], \ldots, \mathrm{E}[X_n])^{tr}$$

is the mean vector of $\vec{X}$ *and*

$$\Sigma := \begin{pmatrix} \mathrm{Cov}(X_1, X_1) & \mathrm{Cov}(X_1, X_2) & \ldots & \mathrm{Cov}(X_1, X_n) \\ \mathrm{Cov}(X_2, X_1) & \mathrm{Cov}(X_2, X_2) & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ \mathrm{Cov}(X_n, X_1) & \ldots & \ldots & \mathrm{Cov}(X_n, X_n) \end{pmatrix}$$

Remark:

The characteristic function can also be defined for a vector $\vec{X} := (X_1, X_2, \ldots, X_n)^{tr}$ of random variables and is given by

$$\varphi_{\vec{X}} : \mathbb{R}^n \to \mathbb{C}$$
$$\vec{t} = (t_1, t_2, \ldots, t_n)^{tr} \mapsto \mathrm{E}\left[\exp\left(i\left\langle \vec{t}, \vec{X}\right\rangle\right)\right] = \mathrm{E}\left[\exp\left(i\sum_{j=1}^{n} t_j X_j\right)\right].$$

One can show that the characteristic function of a random vector $\vec{X} := (X_1, X_2, \ldots, X_n)^{tr}$ with a $\mathcal{N}(\vec{\mu}, \Sigma)$ distribution is given by

$$\varphi_{\vec{X}}(\ \vec{t}\ ) = \exp(i \cdot \vec{t}^{\ tr} \cdot \vec{\mu} - \tfrac{1}{2}\vec{t}^{\ tr} \cdot \Sigma \cdot \vec{t}\ ).$$

Since the distribution of every random vector is uniquely determined by its charateristic function it turns that the normal distribution $\mathcal{N}(\vec{\mu}, \Sigma)$ is completely charaterized by $\vec{\mu}$ and Σ.

Fact 4:

Theorem

Consider a random vector

$$\vec{X} := \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix}$$

with a $\mathcal{N}(\vec{\mu}, \Sigma)$ distribution, some matrix $A \in \mathbb{R}^{k \times n}$ and some vector $\vec{b} \in \mathbb{R}^k$. Then

$$A\vec{X} + \vec{b}$$

has a $\mathcal{N}(A\vec{\mu} + \vec{b}, A\Sigma A^{tr})$-distribution.

Proof: We will show this Theorem in Exercise 2.1!

Fact 5:

Theorem

Consider a random vector $\vec{X} := (X_1, X_2, \ldots, X_n)^{tr}$ *where* $\vec{X} \sim \mathcal{N}(\vec{\mu}, \Sigma)$. *If* Σ *is not singular, i.e.* $det(\Sigma) > 0$ *then the inverse* Σ^{-1} *exists and* $\vec{X}$ *has the density*

$$f_{\vec{X}}(\vec{x}) = \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \cdot \frac{1}{\sqrt{det(\Sigma)}} \cdot \exp\left(-\frac{1}{2}(\vec{x} - \vec{\mu})^{tr} \cdot \Sigma^{-1} \cdot (\vec{x} - \vec{\mu})\right)$$

with respect to the n-dimensional Lebesgue-measure.

Fact 6:

Theorem

The components of a vector $\vec{X}$ with a normal distribution are independent iff they are uncorrelated.

Corollary

The components of a vector $\vec{X}$ with a normal distribution are independent iff they are pairwise independent.

Proof:

We only have to show that uncorrelatedness implies independence: By assumption we have $\mathrm{Cov}(X_i, X_j) = 0$ for $i \neq j$. Therefore the covariance matrix is diagonal with entries $\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2$ which denote the variance of $X_1, X_2, \ldots, X_n$. If one the variances is 0 then that component is degenerate (i.e. constant with probability 1) and hence it is independent of all other components. We can therefore assume w.l.o.g. that we have $\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2 > 0$ which implies that the inverse Σ^{-1} exists and is also diagonal with entries $1/\sigma_1^2, 1/\sigma_2^2, \ldots, 1/\sigma_n^2$. Therefore the density function of $\vec{X}$ simplifies into

$$f_{\vec{X}}(\vec{x}) = \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \cdot \frac{1}{\sqrt{det(\Sigma)}} \cdot \exp\left(-\frac{1}{2}(\vec{x}-\vec{\mu})^{tr} \cdot \Sigma^{-1} \cdot (\vec{x}-\vec{\mu})\right)$$
$$= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sigma_i} \cdot \exp\left(-\frac{(x_i-\mu_i)^2}{2\sigma_i^2}\right)$$

which shows that the joint density factorizes and therefore we can conclude the independence of the components. ∎

Definition (Stopping time - cf. Definition 14.7)

Let $(\Omega, \mathcal{A})$ *be a measurable space and* $(\mathcal{A}_n)_{n \in \mathbb{N}_0}$ *a filtration of* $(\Omega, \mathcal{A})$. *A* **stopping time** *is a function*

$$T : \Omega \to \{0, 1, 2, \ldots\} \cup \{\infty\}$$

such that

$$\{T = n\} \in \mathcal{A}_n$$

for all $n \in \mathbb{N}_0$.

Theorem (The optional stopping theorem - cf. Theorem 14.9)

Let $(\Omega, \mathcal{A}, \mathrm{P})$ *be a probability space and* $(\mathcal{A}_n)_{n\in\mathbb{N}_0}$ *a filtration of* $(\Omega, \mathcal{A})$*:*

1. *If* $(M_n)_{n\in\mathbb{N}_0}$ *is a martingale with respect to* $(\mathcal{A}_n)_{n\in\mathbb{N}_0}$ *and* T *a stopping time, then the stopped martingale*
$$(M_{T\wedge n})_{n\in\mathbb{N}_0}$$
is again a martingale with respect to $(\mathcal{A}_n)_{n\in\mathbb{N}_0}$*.*
2. *If in addition* $\mathrm{P}(T < \infty) = 1$ *and* $(M_{T\wedge n})_{n\in\mathbb{N}_0}$ *is uniformly integrable, then*
$$\mathrm{E}[M_T] = \mathrm{E}[M_0].$$

Sufficient conditions for uniform integrability

Let I be some index set and $(X_i)_{i \in I}$ a family of random variables. Each of the following conditions is sufficient for the uniform integrability of $(X_i)_{i \in I}$:

1. The index set I is finite and we have $X_i \in \mathcal{L}^1 \; \forall i \in I$.
2. There exists a uniform integrable family of random variables $(Y_i)_{i \in I}$ such that $|X_i| \leq |Y_i| \; \forall i \in I$.
3. For some $p > 1$ we have $X_i \in \mathcal{L}^p \; \forall i \in I$.

Bibliography

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An Intermediate Course in Probability.
Springer, 1995.
(*Chapter 5: The Multivariate Normal Distribution*)

Achim Klenke.
Probability Theory.
Springer, 2008.
(*Chapter 6: Convergence Theorems (includes a lot of information about uniform integrability)*)