Stochastic Analysis
Exercise Sheet 2

November 17/November 24, 2011

Exercise 2.1: (Slide 6 from "Exercise Sheet 1")

The characteristic function can also be defined for a vector $\vec{X} := (X_1, X_2, \ldots, X_n)^{tr}$ of random variables and is given by

$$\varphi_{\vec{X}} : \mathbb{R}^n \to \mathbb{C}$$

$$\vec{t} = (t_1, t_2, \ldots, t_n)^{tr} \mapsto \mathrm{E}\left[\exp\left(i\left\langle \vec{t}, \vec{X}\right\rangle\right)\right] = \mathrm{E}\left[\exp\left(i\sum_{j=1}^{n} t_j X_j\right)\right].$$

One can show that the characteristic function of a random vector $\vec{X} := (X_1, X_2, \ldots, X_n)^{tr}$ with a $\mathcal{N}(\vec{\mu}, \Sigma)$ distribution is given by

$$\varphi_{\vec{X}}(\,\vec{t}\,) = \exp(i \cdot \vec{t}^{\,tr} \cdot \vec{\mu} - \tfrac{1}{2}\vec{t}^{\,tr} \cdot \Sigma \cdot \vec{t}\,).$$

Since the distribution of every random vector is uniquely determined by its charateristic function it turns that the normal distribution $\mathcal{N}(\vec{\mu}, \Sigma)$ is completely charaterized by $\vec{\mu}$ and Σ.

Exercise 2.1: (Slide 8 from "Exercise Sheet 1")

Theorem

Consider a random vector $\vec{X} := (X_1, X_2, \ldots, X_n)^{tr}$ *where* $\vec{X} \sim \mathcal{N}(\vec{\mu}, \Sigma)$. *If* Σ *is not singular, i.e.* $det(\Sigma) > 0$ *then the inverse* Σ^{-1} *exists and* $\vec{X}$ *has the density*

$$f_{\vec{X}}(\vec{x}) = \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \cdot \frac{1}{\sqrt{det(\Sigma)}} \cdot \exp\left(-\frac{1}{2}(\vec{x} - \vec{\mu})^{tr} \cdot \Sigma^{-1} \cdot (\vec{x} - \vec{\mu})\right)$$

with respect to the n-dimensional Lebesgue-measure.

Exercise 2.2:

Warning!
Exercise 2.2 shows that the sum $X + Y$ of two random variables X and Y with a (one-dimensional) normal distribution does not always have a normal distribution itself!
But: In the case in which (X, Y) has a (multivariate) normal distribution we know that $X + Y$ also has a normal distribution. (Most important case which is included: X and Y normally distributed and independent!)

Exercise 2.3:

Definition

A stochastic process $(B_t)_{0\le t\le 1}$ *is a Brownian motion if*

1. $B_0 = 0 \quad a.s.$
2. *For* $0 \le t_0 < t_1 < t_2 < \ldots < t_n \le 1$ *the increments* $B_{t_i} - B_{t_{i-1}}$ *(for* $i = 1, 2, \ldots, n$*) are independent with law* $\mathcal{N}(0, t_i - t_{i-1})$.
3. *The mapping*
$$t \mapsto B_t(\omega)$$
is continuous (on $[0,1]$*) for* P*-a.a.* ω.

Exercise 2.4:

The sentence

"Let $(B_t)_{0\leq t\leq 1}$ be a Brownian motion..."

is not as harmless as it appears. But we have succesfully shown the existence of the Brownian motion in the lecture.

Exercise 2.4:

With the help of our constructed **Wiener measure** it is now easy to explicitely construct a probability space $(\Omega, \mathcal{A}, \mathrm{P})$ on which a Brownian motion $(B_t)_{0 \leq t \leq 1}$ exists. Choose

$$\begin{aligned} \Omega &:= C[0,1] \\ \mathcal{A} &:= \mathcal{F} \\ \mathrm{P} &:= \text{"Wiener measure" on } C[0,1] \end{aligned}$$

where $\mathcal{F} = \sigma\left(\left(g_t\right)_{0 \leq t \leq 1}\right)$ (like in Exercise 2.4). Then

$$\left(g_t\right)_{0 \leq t \leq 1}$$

is a Brownian motion (under P).

Exercise 2.4:

But is the distribution of the Brownian motion really uniquely determined by our three defining properties?

Answer: Yes!

Theorem

The distribution of the Brownian motion (i.e. the Wiener measure) is uniquely determined by its finite-dimensional marginal distributions.

Remark:
The finite-dimensional marginal distributions of the Brownian motion $(B_t)_{0 \leq t \leq 1}$ are the distributions of

$$(B_{t_1}, B_{t_2}, \ldots, B_{t_n})$$

for all choices $0 \leq t_0 < t_1 < t_2 < \ldots < t_n \leq 1$ and these distributions are given by the defining properties of the Brownian motion (see also Exercise 3.1!).

Exercise 2.4:

Proof: Exercise 2.4 shows

$$\mathcal{F} = \sigma(\mathcal{Z})$$

where $\mathcal{Z}$ is the collection of all cylinder sets. Note that $\mathcal{Z}$ is stable under intersections. Therefore every probability measure $\mathbb{P}$ on $(C[0,1], \mathcal{F})$ is uniquely determined by the values of $\mathbb{P}(Z)$, $Z \in \mathcal{Z}$. And $\mathbb{P}(Z)$ only depends on finitely many time points for all sets $Z \in \mathcal{Z}$.