Stochastic Analysis
Exercise Sheet 4

December 15

The Cameron-Martin-Theorem

Theorem

Consider a Brownian motion $(B_t)_{t\in[0,1]}$ *with continuous paths. Let* μ *denote the distribution of* $(B_t)_{t\in[0,1]}$ *on* $(C[0,1],\mathcal{F})$ *(i.e.* μ *is the Wiener measure) and let* μ_h *denote the distribution of* $(B_t+h(t))_{t\in[0,1]}$ *for some function* $h\in C[0,1]$*. Then we have:*

1. *If* $h\notin\mathcal{H}$*, then* $\mu_h\perp\mu$*.*
2. *If* $h\in\mathcal{H}$*, then* $\mu_h\approx\mu$*.*

Thereby

$$\mathcal{H}:=\Big\{h\in C[0,1]:\ \text{There is } f\in\mathcal{L}^2[0,1] \text{ such that } h(t)=\int_0^t f(s)ds,\ 0\le t\le 1\Big\}$$

denotes the Cameron-Martin space.

Singular measures

Recall:

$$\mu_h \perp \mu,$$

i.e. μ and μ_h are singular, if there is at least one set $A \in \mathcal{F}$ such that

$$\mu(A) = 1 \quad \text{and} \quad \mu_h(A^c) = 1.$$

Remember

In order to show $\mu_h \perp \mu$, it is enough to find **one set** *with this total different behaviour under μ_h and μ.*

Therefore the total mass of the two probability measure lies on the disjoint sets A and A^c, i.e. the information of μ cannot be used to characterize μ_h.

Example: The distributions of $(B_t)_{t \in [0,1]}$ and $(B_t + x)_{t \in [0,1]}$ are singular for $x \neq 0$.

Equivalent measures

Recall:

$$\mu_h \approx \mu,$$

i.e. μ and μ_h are equivalent, if $\mu \ll \mu_h$ and $\mu_h \ll \mu$. Here $\mu \ll \mu_h$ means that μ is absolute continuous with respect to μ_h, i.e. we have for all $A \in \mathcal{F}$

$$\begin{aligned} \mu_h(A) = 0 \quad &\Rightarrow \quad \mu(A) = 0 \quad \text{and} \\ \mu_h(A) = 1 \quad &\Rightarrow \quad \mu(A) = 1. \end{aligned}$$

Remember

$\mu_h \approx \mu$ has the consequence that all sets with probability 0 or 1 have this property under both probability measures μ_h and μ.

Absolute continuity and the Radon-Nikodym theorem

Theorem

We have $\mu \ll \mu_h$ iff there exists a density f such that

$$\mu(A) = \int_A f(x) d\mu_h(x)$$

for all $A \in \mathcal{F}$. We denote this density f by $\frac{d\mu}{d\mu_h}$.

Proof: See Theorem 9.8 (Radon-Nikodym Theorem) of Probability Theory!

Importance of equivalent measures

For probabilists:
The Cameron-Martin Theorem helps us to characterize the paths of $(B_t + h(t))_{t\in[0,1]}$ for $h \in \mathcal{H}$:

Differentiability:

$$\mathrm{P}\big(B_t + h(t) \text{ is differentiable for some } t_0 \in [0,1]\big) = 0$$

Quadratic variation:

$$\mathrm{P}\big(\big\langle (B_t + h(t))_{t\in[0,1]}\big\rangle_T = T \text{ for all T } \in [0,1]\big) = 1$$

(where $\langle\cdot\rangle_T$ denotes the quadratic variation in T)

(**Remark:** This statement about the quadratic variation is a little stronger than Theorem 2.11 from the lecture. The missing part can be proven by using that $\langle\cdot\rangle_T$ is monotone increasing in T.)

Importance of equivalent measures

For financial mathematics:

A short outlook on continuous models in financial mathematics

In financial mathematics the models which are used are often determined by stochastic differential equations. For example in the generalized Black-Scholes Model the following equation appears

$$dX(t) = \alpha(t,\omega)X(t)dt + \beta(t,\omega)X(t)dB(t)$$

where $\big(B(t)\big)_{t\in[0,1]}$ is a Brownian motion.

We will see that the solution is given by

$$X(t) = X(0)\exp\left(\int_0^t \beta(s,\omega)dB(s) + \int_0^t \left(\alpha(s,\omega) - \tfrac{1}{2}\beta^2(s,\omega)\right) ds\right)$$

So there is a lot of similarity with our density in Exercise 4.2 (ii). Coincidence?

Answer: No!

In financial mathematics the search for equivalent (martingale) measures plays a key role. In the generalized Black-Scholes Model there exists a function

$$u(t,\omega)$$

which depends on the parameters $\alpha(t,\omega), \beta(t,\omega)$ (and the interest rate) of the model. Under some conditions on $u(t,\omega)$ there exists an equivalent martingale measure Q which is determined by the density

$$\frac{dQ}{dP}(\omega) = \exp\left(-\int_0^1 u(t,\omega)dB(s) - \frac{1}{2}\int_0^1 u^2(s,\omega)ds\right)$$

(where we consider the time interval $[0,1]$ here). Martingale measure means here that the process $(X(t))_{t\in[0,1]}$ from the last slide is a martingale with respect to the new measure Q.

The Girsanov Theorem

Interesting consequence of the change of measure:
The change of measure eliminates the drift! More precisely, we have for a Brownian motion $(B_t)_{t\in[0,1]}$ and some function $h \in \mathcal{H}$ that the Brownian motion $(\widetilde{B}_t)_{t\in[0,1]}$ with drift defined by

$$\widetilde{B}_t := B_t + h(t)$$

is a standard Brownian motion (without drift!) with respect to the measure P_{-h} which is determined by the density

$$\frac{dP_{-h}}{dP} = \exp\left(-\int_0^1 h' dB(s) - \tfrac{1}{2}\int_0^1 h'(s)^2 ds\right).$$

Remark: We hope to prove this statement at the end of the lecture in the broader setting of the Girsanov Theorem.

Exercise 4.2

Exercise 4.2: The mathemetical construction behind it!

The Wiener measure μ

For any Brownian motion which is constructed on some probability space $(\Omega, \mathcal{A}, \mathrm{P})$ there exists a set $A \in \mathcal{A}$ with $\mathrm{P}(A) = 1$ and

$$t \mapsto B_t(\omega)$$

is continuous (in t) for $\omega \in A$. In particular the mapping $\left(\widetilde{B}_t\right)_{t\in[0,1]}$

$$\left(\widetilde{B}_t\right)_{t\in[0,1]} : (\Omega, \mathcal{A}) \to (C[0,1], \mathcal{F})$$

$$\omega \mapsto \begin{cases} \left(B_t(\omega)\right)_{t\in[0,1]} & \text{for } \omega \in A \\ (0)_{t\in[0,1]} & \text{for } \omega \in A^c \end{cases}$$

is well defined and measurable. Note that $\left(\widetilde{B}_t\right)_{t\in[0,1]}$ and $\left(B_t\right)_{t\in[0,1]}$ only differ on a set of probability 0!

The Wiener measure μ and the measure μ_h

To avoid this technical problem we will assume for the rest of the slides that the mapping

$$t \mapsto B_t(\omega)$$

is continuous for all ω! Then $\left(B_t\right)_{t\in[0,1]}$ really is a random variable with values in $(C[0,1], \mathcal{F})$ and therefore it induces a probability measure μ on $(C[0,1], \mathcal{F})$ - which is the image measure under P (cf. Lemma 2.8 in Probability Theory). This measure is called Wiener measure and has the property

$$\mu(A) = \mathsf{P}^{\left(B_t\right)_{t\in[0,1]}}(A) = \mathsf{P}\left(\left(B_t\right)_{t\in[0,1]} \in A\right) \quad \text{for } A \in \mathcal{F}.$$

We can use the same construction to get the distribution μ_h of the Brownian motion with drift for $h \in C[0,1]$:

$$\mu_h(A) = \mathsf{P}^{\left(B_t + h(t)\right)_{t\in[0,1]}}(A) = \mathsf{P}\left(\left(B_t + h(t)\right)_{t\in[0,1]} \in A\right) \quad \text{for } A \in \mathcal{F}.$$

The measures μ_n and $\mu_{h,n}$

Recall that μ and μ_h are measures on $(C[0,1],\mathcal{F})$. We can now consider the following sub-σ-fields $\mathcal{F}_n$ of $\mathcal{F}$

$$\mathcal{F}_n := \sigma\big(\{g_t : \ t \in \mathcal{D}_n\}\big) \quad \text{where } \mathcal{D}_n := \{j2^{-n} : \ j = 0, 1, \ldots, 2^n\}$$

where by definition

$$\mathcal{F} := \sigma\big(\{g_t : 0 \leq t \leq 1\}\big).$$

Then

$$\mu_n := \mu\Big|_{\mathcal{F}_n} \quad \text{and} \quad \mu_{h,n} := \mu_h\Big|_{\mathcal{F}_n}$$

are the restriction of μ_n and $\mu_{h,n}$ to $\mathcal{F}_n$ in which the sets only depend on finitely many time points $t \in \{j2^{-n} : \ j = 0, 1, \ldots, 2^n\}$. More precisely for every $A \in \mathcal{F}_n$ there exists $C \in \mathcal{B}(\mathbb{R}^{2^n})$ with

$$A = \Big((g_t)_{t\in\mathcal{D}_n}\Big)^{-1}(C).$$

Overview:

In order to work with μ and μ_h the following picture might help:

$$(\Omega, \mathcal{A}) \longrightarrow (C[0,1], \mathcal{F})$$
$$P \xrightarrow{(B_t)_{t\in[0,1]}} \mu$$
$$P \xrightarrow{(B_t+h(t))_{t\in[0,1]}} \mu_h$$

Further we have the following connection of μ_n and $\mu_{h,n}$ with the finite-dimensional marginal distributions:

$$(C[0,1], \mathcal{F}_n) \longrightarrow (\mathbb{R}^{2^n}, \mathcal{B}(\mathbb{R}^{2^n}))$$
$$\mu_n \xrightarrow{(g_t)_{t\in\mathcal{D}_n}} P^{(B_t)_{t\in\mathcal{D}_n}}$$
$$\mu_{h,n} \xrightarrow{(g_t)_{t\in\mathcal{D}_n}} P^{(B_t+h(t))_{t\in\mathcal{D}_n}}$$

Thereby $\mathcal{F}_n = \left((g_t)_{t\in\mathcal{D}_n}\right)^{-1}\left(\mathcal{B}(\mathbb{R}^{2^n})\right)$. Further μ and μ_h are uniquely determined by $(\mu_n)_{n\in\mathbb{N}}$ and $(\mu_{h,n})_{n\in\mathbb{N}}$ respectively!