Stochastic Analysis
Exercise Sheet 5

January 12

The optional stopping theorem

There are many different versions of the optional stopping theorem for martingales in continuous time. Usually it is easier to consider stopping times S, T which are bounded (by some constant):

Theorem (Optional stopping theorem for bounded stopping times)

Suppose $(X_t)_{t\geq 0}$ is a continuous martingale with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$ and $0 \leq S \leq T$ are bounded stopping times, i.e. there exists $K \in \mathbb{R}_+$ such that $\mathrm{P}(T \leq K) = 1$. Then

$$\mathrm{E}[X_T|\mathcal{F}_S] = X_S \quad a.s.$$

In particular

$$\mathrm{E}[X_T] = \mathrm{E}[\mathrm{E}[X_T|\mathcal{F}_S]] = \mathrm{E}[X_S].$$

Remark: In many applications this theorem is applied to some bounded stopping time T, $S = 0$ and $X_0 = 0$. Then we have

$$\mathrm{E}[X_T] = \mathrm{E}[X_0] = 0.$$

The optional stopping theorem

If we want to consider arbitrary stopping times we need some additional condition, for example a condition on the stopped martingale:

Theorem (Theorem 4.8)

Suppose $(X_t)_{t\geq 0}$ is a continuous martingale with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$ and $0 \leq S \leq T$ are stopping times. If the process $(X_{t\wedge T})_{t\geq 0}$ is dominated by an integrable random variale X, i.e.

$$|X_{t\wedge T}| \leq X \quad a.s., \forall t \geq 0,$$

then

$$\mathrm{E}[X_T|\mathcal{F}_S] = X_S \quad a.s.$$

In particular

$$\mathrm{E}[X_T] = \mathrm{E}[\mathrm{E}[X_T|\mathcal{F}_S]] = \mathrm{E}[X_S].$$

The optional stopping theorem

Remark 1: The condition

$$|X_{t\wedge T}| \leq X \quad \text{a.s., } \forall t \geq 0,$$

for some integrable random variable X implies that $(X_{t\wedge T})_{t\geq 0}$ is uniformly integrable (for a proof (in the discrete case) see also the Lemma in the uploaded solution of Sheet 1 and compare slide 15 of Sheet 1).

Remark 2: Note also that we do not have to assume

$$\mathrm{P}(T < \infty) = 1$$

if we use the usual convention

$$X_T = \begin{cases} X_T & \text{on } \{T < \infty\} \\ \lim\limits_{t\to\infty} X_t & \text{on } \{T = \infty\}. \end{cases}$$

The optional stopping theorem

Thereby

$$\lim_{t\to\infty} X_t \text{ is well defined on } \{T=\infty\}$$

due to the uniform integrability of $(X_{t\wedge T})_{t\geq 0}$ since the following theorem holds (use $Y_t := X_{t\wedge T}$):

Theorem (Martingale closure theorem)

Suppose $(Y_t)_{t\geq 0}$ is a continuous martingale with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$. Then the following two conditions are equivalent:

1. *$(Y_t)_{t\geq 0}$ is uniformly integrable*
2. *There exists an integrable random variable Y_∞ such that $Y_t = E[Y_\infty|\mathcal{F}_t]$ a.s.*

Under these conditions we further have that $(Y_t)_{t\geq 0}$ converges a.s. (and in $\mathcal{L}^1$).

Remark: The limit of $\lim_{t\to\infty} Y_t$ is given by

$$\mathrm{E}[Y_\infty|\mathcal{F}_\infty] \quad \text{where } \mathcal{F}_\infty := \sigma\Big(\Big\{\mathcal{F}_t :\ t\geq 0\Big\}\Big).$$

Different filtrations of a (sub-/super-)martingale:

Lemma

If a process $(M_t)_{t\geq 0}$ is a (sub-/super-)martingale with respect to any filtration $(\mathcal{G}_t)_{t\geq 0}$, then $(M_t)_{t\geq 0}$ is also a (sub-/super-)martingale with respect to its own filtration $(\mathcal{F}_t)_{t\geq 0}$, where

$$\mathcal{F}_t := \sigma\big(\{M_s : \ 0 \leq s \leq t\}\big).$$

Proof: Since M_t has to be measurable with respect to $\mathcal{G}_t$ for all $t \geq 0$ by assumption, we have $\mathcal{F}_t \subseteq \mathcal{G}_t$ for all $t \geq 0$. Therefore the projection property of the conditional expectation implies for $t > s$

$$\mathrm{E}[M_t|\mathcal{F}_s] = \mathrm{E}[\mathrm{E}[M_t|\mathcal{G}_s]\mathcal{F}_s] \begin{array}{c}\geq \\ = \\ \leq\end{array} \mathrm{E}[M_s|\mathcal{F}_s] = M_s$$

due to the monotonicity of the conditional expectation.

Bibliography

D. Revuz, Marc Yor.
Continuous martingales and Brownian motion.
Springer, 2005.
(*Chapter II.3: The optional stopping theorem*)