Exercise sheet 1
Hand in: Wednesday, 29.10.2014, 12:00-14:00

Definition. A function $f : \mathbb{R} \to \mathbb{R} \cup \{-\infty, \infty\}$ is called upper semicontinuous at $x_0 \in \mathbb{R}$ if

$$\limsup_{x \to x_0} f(x) \leq f(x_0).$$

Similarly, a function $f : \mathbb{R} \to \mathbb{R} \cup \{-\infty, \infty\}$ is called lower semicontinuous at $x_0 \in \mathbb{R}$ if

$$\liminf_{x \to x_0} f(x) \geq f(x_0).$$

Exercise 1 [5 Points].

(i) Determine whether the following functions are upper/lower semicontinuous and justify your statement:

(a) $f = 1_G$, where $G \subset \mathbb{R}$ is open.

(b) $f = 1_C$, where $C \subset \mathbb{R}$ is closed.

(c) $f(x) = \lfloor x \rfloor$ for all $x \in \mathbb{R}$.

(d) $f(x) = \lceil x \rceil$ for all $x \in \mathbb{R}$

(ii) Give an example of a lower semicontinuous function which is neither left nor right continuous.

(iii) Show that a function f is lower semicontinuous if and only if, for all $\alpha \in \mathbb{R}$, the set $A_\alpha = \{x : f(x) \leq \alpha\}$ is closed. $A(\alpha)$ is called the *level set* of the function f.

(iv) Let $\{f_i\}_{i \in I}$ be a family of lower semicontinuous functions, where I is some non-empty index set. Show that the pointwise supremum, defined by,

$$f(x) = \sup_{i \in I} f_i(x)$$

is also lower semicontinuous.

(v) Let K be a compact subset of the real line and $f : K \to (-\infty, \infty]$ a lower semicontinuous function on K. Prove that f attains a minimum on K.

Exercise 2 [3 Points].
Let $\Lambda(\cdot)$ be a smooth and convex function on $\mathbb{R}$ such that

$$\Lambda^\star(x) = \sup_{\lambda \in \mathbb{R}} \Big\{ \lambda x - \Lambda(\lambda) \Big\} \quad x \in \mathbb{R},$$

is finite for all $x \in \mathbb{R}$. Then, show that

$$\Lambda(\lambda) = \sup_{x \in \mathbb{R}} \Big\{ \lambda x - \Lambda^\star(x) \Big\}.$$

Please turn!

Exercise 3 [5 Points].
Let X be a standard Gaussian random variable (i.e, $X \sim \mathcal{N}(0,1)$). Show that

$$\frac{1}{\sqrt{2\pi}}\Big(\frac{a}{1+a^2}\Big)e^{-a^2/2} \leq \mathbb{P}(X > a) \leq \frac{1}{\sqrt{2\pi}}\frac{1}{a}e^{-a^2/2}.$$

We note that this estimate is even better than the Cramér bound, though valid only for Gaussians.

Exercise 4 [7 Points].
Give an example of a sequence of independent and identically distributed (i.i.d) random variables $(X_n)_{n\in\mathbb{N}}$ and a Borel set $A \subset \mathbb{R}$ so that

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\Big(\frac{S_n}{n} \in A\Big)$$

fails to exist. Here $S_n = X_1 + \ldots X_n$.