Exercise sheet 4

Hand in: Wednesday, 10.12.2014, 12:00-14:00

Exercise 1 [10 Points].
Let E and $\widetilde{E}$ be Polish spaces and $(\mu_n)_n$ a sequence in $\mathcal{M}_1(E)$ which satisfies a large deviation principle with a good rate function I. Let $\Phi : E \longrightarrow \widetilde{E}$ be a map and $(F_L)_{L\in\mathbb{N}}$ an increasing sequence of closed subsets of E so that $F = \cup_L F_L$ with $\mu_n(F) = 1$ and the following hold true:

- $\limsup_{n\to\infty} \frac{1}{n} \log \mu_n(F_L^c) \leq -L \quad \forall L \in \mathbb{N}$.
- The restrictions $\Phi\Big|_{F_L}$ are continuous for all $L \in \mathbb{N}$.

We set $\widetilde{\mu}_n = \mu_n \circ \Phi^{-1}$ and $\widetilde{I} : \widetilde{E} \longrightarrow \mathbb{R} \cup \{\infty\}$, $\widetilde{I}(\tilde{x}) = \inf \Big\{ I(x) \colon x \in F, \Phi(x) = \tilde{x} \Big\}$. Show that the sequence $(\widetilde{\mu}_n)_n$ satisfies a large deviation principle with a good rate function $\widetilde{I}$.

Exercise 2 [10 Points].
Let ν be a probability measure on $(\mathbb{R}^{\mathbb{N}}, \mathcal{B}^{\mathbb{N}})$ and let μ be a product measure on $(\mathbb{R}^{\mathbb{N}}, \mathcal{B}^{\mathbb{N}})$ with marginal distributions $(\mu_n)_{n\in\mathbb{N}}$. Furthermore, suppose that $(\nu_n)_{n\in\mathbb{N}}$ is the n-th marginal distribution on of ν and η the infinite product measure with marginal distributions $(\nu_n)_{n\in\mathbb{N}}$. In other words, $\nu_n = \nu \circ \pi_n^{-1}$ with $\pi_n : \mathbb{R}^{\mathbb{N}} \longrightarrow \mathbb{R}$, $(x_i)_{i\in\mathbb{N}} \mapsto x_n$ and $\eta = \prod_{n\in\mathbb{N}} \nu_n$. Show that the following implication holds:

$$H(\nu|\mu) < \infty \Rightarrow \nu \ll \eta.$$

Exercise 4 [10 Points]. Let $C_0([0,1])$ be the space of continuous functions vanishing at the origin and let $\mathbb{P}$ denote the Wiener measure corresponding to the standard Brownian motion $B = (B_t)_{0\leq t\leq 1}$. Show that, for any $p > 1$,

$$\lim_{\lambda\to\infty} \frac{1}{\lambda^2} \log \mathbb{P}\Big\{ \int_0^1 dt\, |B(t)|^p \geq \lambda^p \Big\} = - \inf_{\|f\|_p \geq 1} I(f)$$

where

$$I(f) = \begin{cases} \frac{1}{2} \int_0^1 (f'(t))^2 dt & f \in H, \\ \infty & \text{else}, \end{cases}$$

where you can recall (from the lecture), that H is the space of absolutely continuous functions with square integrable derivatives.