Exercise sheet 5

Hand in: Wednesday, 7.1.2015, 12:00-14:00

Exercise 1: [10 points]
Let $(Y_n)_{n\in\mathbb{N}}$ be a sequence of independent and identically distributed random variables with common distribution

$$\mu = \frac{1}{2}\delta_{1/2} + \frac{1}{2}\delta_{3/2}.$$

Let $S_n = X_1 + \cdots + X_n$ denote the partial sum. Prove that

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{E}\bigg\{\bigg(\frac{S_n}{n}\bigg)^n\bigg\}$$

exists and is strictly positive.

Exercise 2: [10 points]
Let μ be the uniform distribution on a finite set $\{a_1, \dots, a_n\}$, $n \in \mathbb{N}$. Find a probability measure ν supported on the set $\{a_1, \dots, a_m\}$ such that $m < n$ and ν minimizes the relative entropy $H(\nu|\mu)$.

Exercise 3 [10 points]
Let $(\Omega, \mathcal{F})$ be a measurable space and $\mu, \nu \in \mathcal{M}_1(\Omega, \mathcal{F})$. Let $\mathcal{F}_0, \mathcal{F}_1$ be sub-σ algebras of $\mathcal{F}$ so that $\mathcal{F}_0 \subset \mathcal{F}_1$. The restrictions on the relative entropy on $\mathcal{F}_0$ and $\mathcal{F}_1$ are defined

$$H(\mu|\nu)|_{\mathcal{F}_j} = H\Big(\mu|_{\mathcal{F}_j} \;\Big|\; \nu|_{\mathcal{F}_j}\Big) \qquad j = 0, 1,$$

where $\mu|_{\mathcal{F}_j}$ is the restriction of μ to $\mathcal{F}_j$. Show that the following monotonicity holds:

$$H(\mu|\nu)\Big|_{\mathcal{F}_0} \le H(\mu|\nu)\Big|_{\mathcal{F}_j}.$$

Exercise 4 [10 points]
Find a sequence $(a_n)_{n\ge1}$ such that $\limsup_{n\to\infty} \frac{1}{a_n}\int_0^n B_s^2 ds$ is finite and strictly positive, where $(B_s)_{s\ge0}$ is a Brownian motion.