Exercise-1 :

(i) $\mu_n \sim \text{unif}([e^{-n}, e^{n}])$ easy.

(ii) $\mu_n = \text{Pois}(\frac{\lambda}{n})$ easy.

(iii) $\mu_n = (1-\frac{1}{n})\delta_0 + \frac{1}{n}\delta_n \Rightarrow \delta_0$.

Take $I(x) = \begin{cases} 0 & x=0 \\ \infty & \text{else} \end{cases}$

I is a good rate function.

Lower bound: Take G open & $0 \notin G$. Then $\underline{\lim}_{n\to\infty} \frac{1}{n}\log \mu_n(G) \geq -\inf_{x\in G} I(x) = -\infty$

Let $0 \in G$. $\Rightarrow \mu_n(G) \geq 1-\frac{1}{n} \Rightarrow \underline{\lim} \frac{1}{n}\log \mu_n(G) \geq -\inf_{x\in G} I(x) = 0$.

Weak upper bound : K compact. & $0 \notin K$. $\Rightarrow \mu_n(K) = 0$ for n large as K is bounded. $\Rightarrow \overline{\lim_n} \frac{1}{n}\log \mu_n(K) = -\infty = -\inf_{x\in K} I(x)$

$0 \in K \Rightarrow \mu_n(K) = 1-\frac{1}{n}$, n large $\Rightarrow \overline{\lim_n} \frac{1}{n}\log \mu_n(K) = 0 = -\inf_{x\in K} I(x)$

$\Rightarrow$ Weak LDP.

Take $F = [1,\infty)$ closed, but not compact.

But ~~upp~~ Check that upper bound does not hold. No Strong LDP.

Exercise 2 :

This is simple:

A convex. This means, if

$\frac{S_n}{n} \in A$. and $\frac{S_{n+m} - S_n}{n+m-n} \in A$, then

$$\underbrace{\frac{n}{m+n} \cdot \frac{S_n}{n} + \frac{m}{m+n} \cdot \frac{S_{n+m} - S_n}{n+m-n}}_{\parallel} \in A$$

$$\underbrace{\frac{S_n}{m+n} + \frac{S_{n+m} - S_n}{m+n}}_{\parallel} \in A$$

$$\frac{S_{n+m}}{m+n} \in A$$

Now use independence of $\frac{S_n}{n}$ and $\frac{S_{n+m} - S_n}{n+m-n}$ to get the subadditive bound.

Exercise 3 :

By Cramér upper bound, we know

$$\mathbb{P}\left(\frac{S_n}{n} \in [x, \infty)\right) \leq 2\, e^{-n \inf\limits_{y \in [x,\infty)} \Lambda^*(y)} .$$

But since Λ^* is a convex function and $x > m = \mathbb{E}(X_1)$, we can show $\inf\limits_{y \in [x,\infty)} \Lambda^*(y) > 0$. [See solution of Exercise-1 in Exercise Sheet 3 for details.]

This justifies the exponential decay

$$\limsup_{n \to \infty} \frac{1}{n} \log \mathbb{P}\left(\frac{S_n}{n} \in [x, \infty)\right) < 0 .$$

Exercise 4 :

Enough to prove the claim for a standard normal vector $X_1 \sim \mathcal{N}_{d\times d}\left(\underline{0}, I^d_{d\times d}\right)$.

Otherwise, set the usual scaling, by writing, for $i = 1, \dots, d$

$$\left(\tilde{X}_1\right)_i = \frac{(X_1)_i - \mu_i}{\sqrt{|\Sigma|}} .$$

Tail estimate of a d-dimensional Gaussian:

$X \sim \mathcal{N}(\underline{0}, Id_{d\times d})$. Note that, this means

X has mean $\underline{\mu} = (0, \ldots, 0) \in \mathbb{R}^d$ and

covariance matrix $\Sigma = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$ which is

positive definite. In this case, the

density function is

$$f_X(x_1, \ldots, x_d) = \frac{1}{\sqrt{(2\pi)^d |\Sigma|}} \exp\left\{-\frac{1}{2}(\underline{x}-\underline{\mu})^t \Sigma^{-1} (\underline{x}-\underline{\mu})\right\}$$

$$= \frac{1}{(2\pi)^{d/2}} e^{-\frac{1}{2}\|\underline{x}\|^2}$$

$\forall x > 0$, Hence,

So, $$\mathbb{P}(\|X\| > x) = \frac{1}{(2\pi)^{d/2}} \int_{y \in B_x(0)^c} e^{-\frac{1}{2}\|y\|^2} \, dy$$

($y \mapsto e^{-\frac{1}{2}\|y\|^2}$ is a radial fct. we change to polar coordinates) $$= \frac{1}{(2\pi)^{d/2}} \int_x^\infty e^{-\frac{r^2}{2}} r^{d-1} \, dr$$

we know what to do.

For d=1:

For d=2: $\int_x^\infty r e^{-r^2/2} \, dr = \int_{x^2/2}^\infty e^{-s} \, ds = e^{-x^2/2}$

For d≥3: $\int_x^\infty r^{d-2} \cdot r e^{-r^2/2}$ & keep performing integration by parts till r^{d-2} disappear by differentiating. We'll get a term $P(x) e^{-x^2/2}$ where $P(x)$ is a polynomial in x.