Solution to problem 1 : (1)

Given : $\mu_n = \mathbb{P} X_n^{-1} \in M_1(\mathbb{R})$. $\mu_n \sim$ LDP with speed $n^{\beta}$ and rate function $I$ s.t.

$I$ is convex and $\{I = 0\} = x_0 \in \mathbb{R}$.

Fix $\delta > 0$ & take $G = B_\delta(x_0)$ ~~a closed~~ an open ball of radius $\delta > 0$ around $x_0$.

By the LDP upper bound,

$$\limsup_{n\to\infty} n^{\beta} \mu_n(G) \leq -\inf_{x \in G^c} I(x) =: -c$$

$\Rightarrow$ $\forall n$ large enough, $\mu_n(B_\delta(x_0)^c) \leq e^{-n^{\beta} c}$

Now, claim : $c > 0$.

Suppose not. Assume, without loss of generality that $x_0 = 0$. So, $\inf_{x : |x| \geq \delta} I(x) = 0$.

Then, $\exists\, x_2 \in B_\delta(0)^c$ s.t. $I(x_2) < \frac{I(\delta)}{2}$.

Now, $\forall\, 0 < \varepsilon < 1$, $\exists\, x_1 = x_1(\varepsilon)$ s.t. $\delta = \varepsilon x_1 + (1-\varepsilon) x_2$ for some $x_1$ s.t. $x_1 \in B_\delta(0)$.

Then $\varepsilon I(x_1) + (1-\varepsilon) I(x_2) < \varepsilon I(x_1) + (1-\varepsilon) \frac{I(\delta)}{2}$.

But, ~~again~~ ~~Becau~~ for ~~some~~ some $\eta \in (0,1)$

$I(x_1) < \eta I(\delta)$ (because of convexity & $I(0) = 0$).

$\Rightarrow$ $\varepsilon I(x_1) + (1-\varepsilon) I(x_2) < I(\delta)\left[\varepsilon \eta + \frac{1-\varepsilon}{2}\right] < I(\delta)$.

This ~~contrad~~ contradicts the fact that $I$ is convex (since $\delta = \varepsilon x_1 + (1-\varepsilon) x_2$).

Hence,

$$\sum_{n \text{ large}} \mathbb{P}\left( |X_n - x_0| \geq \delta \right) \leq \underbrace{\sum_{n \text{ large}} e^{-n^{\beta} c} \ \& \ c > 0.}_{\circledast < \infty.}$$

$\circledast$ By Borel-Cantelli lemma, we conclude that

$$\circledast \quad X_n \to x_0 \qquad \mathbb{P}\text{- a.s.}$$

Exercise Sheet-3 (Solutions):

~~Aufgabe 2~~ Exercise 2:

(i) Need examples for $\mu, \nu \in M_1(\mathbb{R})$ s.t.

$$H(\mu|\nu) \neq H(\nu|\mu)$$

Take $\mu = \delta_0$, $\nu = \frac{1}{2}(\delta_0 + \delta_1)$

$$\Rightarrow \begin{cases} \mu \ll \nu & \Rightarrow \quad \dfrac{d\mu}{d\nu}(x) = \begin{cases} 2 & \text{for } x = 0 \\ 0 & \text{for } x = 1 \end{cases} \\ \nu \not\ll \mu & \Rightarrow \quad H(\nu|\mu) = +\infty. \end{cases}$$

(ii) $d(\mu,\nu) := \frac{1}{2}\left(H(\mu|\nu) + H(\nu|\mu)\right)$

Clearly $d(\cdot,\cdot) \geq 0$, equals zero iff $\mu = \nu$ & $d(\mu,\nu) = d(\nu,\mu)$

will show that the triangle ineq. fails.

Take $S = \{0,1\}$ & define $\pi_\alpha \in M_1(S)$ by

$$\pi_\alpha = \alpha\delta_0 + (1-\alpha)\delta_1 \quad \text{for } \alpha \in (0,1)$$

$$\frac{d\pi_\alpha}{d\pi_\beta}(x) = \begin{cases} \alpha/\beta & \text{for } x = 0 \\ 1-\alpha/1-\beta & \text{for } x = 1 \end{cases}$$

$$\Rightarrow 2\, d(\pi_\alpha, \pi_\beta) = (\alpha - \beta) \log\left(\frac{\alpha(1-\beta)}{\beta(1-\alpha)}\right) \quad [\text{Check!}].$$

We take $\nu := \pi_{9/10}$, $\mu := \pi_{1/10}$, $\xi = \pi_{1/2}$

$$2\,d(\nu,\mu) = 4/5 \log(81)$$

$$2\,d(\nu,\xi) = 2/5 \log(9)$$

$$2\,d(\xi,\mu) = 2/5 \log(9)$$

$$\Rightarrow d(\nu,\mu) > d(\nu,\xi) + d(\xi,\mu).$$

Triangle inequality fails.

gabe 4 : $X$ Polish & $\nu \in M_1(X)$ & $f \in B_b(X)$.

i.e. $f$ bounded Borel measurable.

Need to prove : $\displaystyle -\log \int e^{-f}\, d\nu = \inf_{\mu \in M_1(X)} \left\{ H(\mu|\nu) + \int f\, d\mu \right\}$

Define, $\nu_f \in M_1(X)$ by setting

$$\boxed{\frac{d\nu_f}{d\mu} = e^{-f} / \mathbb{E}_\nu(e^{-f})} \quad \text{i.e.} \quad \boxed{d\nu_f = \frac{e^{-f}}{\mathbb{E}_\nu(e^{-f})}\, d\mu}$$

$$H(\mu|\nu) = \int \log\left(\frac{d\mu}{d\nu}\right) d\mu$$

$$= \int \log\left(\frac{d\mu}{d\nu_f} \cdot \frac{d\nu_f}{d\nu}\right) d\mu$$

$$= \underbrace{\int \log\left(\frac{d\mu}{d\nu_f}\right) d\mu}_{= H(\mu|\nu_f) \geq 0} + \int \log\left(\frac{d\nu_f}{d\nu}\right) d\mu$$

$$H(\mu|\nu) \geq \int \log\left(\frac{d\nu_f}{d\nu}\right) d\mu.$$

$$= \int \log\left(e^{-f} / \mathbb{E}_\nu(e^{-f})\right) d\mu$$

$$= -\int f \, d\mu - \log \mathbb{E}_\nu(e^{-f})$$

$$= -\int f \, d\mu - \log \int e^{-f} \, d\nu$$

$$\forall \mu \in M_1(X)$$

$$\Rightarrow \inf_{\mu \in M_1(X)} \left\{ H(\mu|\nu) + \int f \, d\mu \right\} \geq -\log \int e^{-f} \, d\nu$$

For the inequality in the other direction, take $\mu \in M_1(X)$ such that

$$\frac{d\mu}{d\nu} = e^{-f} / \mathbb{E}_\nu(e^{-f}) \quad (*)$$

(Note that $f \in B_b(X)$ is fixed).

$$\Rightarrow H(\mu|\nu) = -\int f \, d\mu$$

$$\Rightarrow H(\mu|\nu) + \int f \, d\mu = 0.$$

On the other hand, for such a $\mu \in M_1(X)$ as in $(*)$,

$$-\log \int \frac{d\mu}{d\nu} \, d\nu = 0$$

$$= -\log \int e^{-f} \, d\nu.$$

Clearly, the equality holds & the infimum is achieved at $\mu \in M_1(X)$ with $\boxed{\frac{d\mu}{d\nu} = e^{-f}} / \mathbb{E}_\nu(e^{-f})$

**Solution 4.** There is more than one way to do this. One way is as follows. Note that, we can restrict the infimum to all $\nu << \mathcal{N}(0,1)$ (otherwise the relative entropy is infinity). Let $\lambda$ be the Lebesgue measure on $\mathbb{R}$. Then

$$\begin{aligned} H(\nu|N(0,1)) &= \int \log\Big(\frac{d\nu}{dN(0,1)}\Big)d\nu \\ &= H(\nu|\lambda) - \int \log\Big(\frac{dN(0,1)}{d\lambda}\Big)d\nu \\ &\geq \frac{1}{2}\int x^2\nu(dx) \qquad \text{by non-negativity of entropy and some calculation} \\ &\geq \frac{1}{2}\Big(\int x\nu(dx)\Big)^2 \qquad \text{by Jensen.} \end{aligned}$$

To prove the other bound, we indeed show that the infimum is attained at some $\nu$ with mean $z \in \mathbb{R}$. We simply take

$$\nu = \mathcal{N}(z,1)$$

Check that,

$$\frac{dN(z,1)}{dN(0,1)}(x) = \exp\{xz - \frac{1}{2}z^2\}$$

leading to

$$H(\nu|N(0,1)) = \frac{1}{2}z^2.$$