Part-2 Sheet 4:

Need to show: $\lim_{\lambda\to\infty} \frac{1}{\lambda^2} \log \mathbb{P}\left\{ \int_0^1 |B(s)|^p ds \geq \lambda^p \right\} = - \inf_{\|f\|_p \geq 1} I(f)$.

Take $\frac{1}{\lambda^2} = \varepsilon$.

Then, by Schilder's theorem,

$$\underbrace{- \inf_{f\in A^0} I(f) \leq \varliminf_{\varepsilon\to 0} \varepsilon \log \mu_\varepsilon(A) \leq \varlimsup_{\varepsilon\to 0} \varepsilon \log \mu_\varepsilon(A) \leq - \inf_{f\in \bar{A}} I(f)}_{(*)}$$

where μ_ε is the distribution of $\sqrt{\varepsilon}\, B(\cdot)$ and

$$A = \left\{ f \in C_0([0,1]) : \|f\|_p \geq 1 \right\}.$$

Remark: We are working with the sup-norm topology (ie $\forall f \in C_0([0,1])$, $\|f\|_\infty := \sup_{t\in[0,1]} |f(t)|$.) and A^0 and $\bar{A}$ are then defined w.r.t this topology.

Claim 1: A is closed. Why? Take (f_n) in A s.t. $\|f_n - f\|_\infty \to 0$.

$\cancel{\|f\|_\infty = \|f_n - (f_n - f)\|_\infty \geq \|f_n\|_\infty - \|f_n - f\|_\infty \geq \|f_n\|_p - \|f_n - f\|_\infty \geq (1-\varepsilon)\ \forall \varepsilon > 0\ \&\ n \text{ large enough.}}$

Then, $\|f\|_p = \| f_n - (f_n - f) \|_p$

$\geq \|f_n\|_p - \|f_n - f\|_p$

$\geq 1 - \|f_n - f\|_\infty$

$\geq 1 - \varepsilon \quad \forall \varepsilon > 0\ \&\ n$ large enough.

$\Rightarrow \|f\|_p \geq 1$.

~~Claim~~ ~~By~~ By (*), it is enough to prove (2)

$\inf_{f \in A} I(f) = \inf_{f \in A^0} I(f)$. (This shows that there is in fact equality in (*))

Now, $I(f) = \begin{cases} \frac{1}{2}\int_0^1 f'(s)^2\, ds & f \in H^1 \\ \infty & \text{else} \end{cases}$

To minimize $I(f)$ we need to set its (functional) derivative to zero: i.e. take any ~~test~~ smooth test function $\varphi \in C_0^\infty([0,1])$ (i.e. φ is smooth ~~is~~ and vanishes at 0 & 1)

and set $\displaystyle\lim_{\delta \to 0} \frac{I(f+\delta\varphi) - I(f)}{\delta} = 0.$

Note that, we only need to check this for $f \in H^1$ (we are minimizing!).

$$\Leftrightarrow \lim_{\delta \to 0} \frac{\int_0^1 (f'+\delta\varphi')^2 - \int (f')^2}{2\delta} = 0$$

$$\Leftrightarrow \lim_{\delta \to 0} \frac{2\delta \int f'\varphi' + \delta^2 \int \varphi'^2}{\delta} = 0$$

$$\Leftrightarrow \underbrace{\int f'\varphi'}_{= \underbrace{f'\varphi\big|_0^1}_{=0} - \int f''\varphi} = 0$$

(as φ has zero boundary condition)

$$\Leftrightarrow \int f''\varphi = 0$$

$$\Leftrightarrow f'' \equiv 0$$

(Since in the Hilbert space any element can be uniquely determined by its inner product)

This means, $f'(s) = c$ (some constant)

$\Rightarrow f(s) = cs + c'$. But $c' = 0$ (as $f(0) = 0$)

$\Rightarrow f(s) = cs$

$$\Rightarrow \inf_{f \in A} I(f) = \frac{1}{2} \inf_{\|f\|_p \geq 1} \int_0^1 f'(s)^2 \, ds$$

$$= \frac{1}{2} \inf_{c \geq (1+p)^{1/p}} c^2 \quad \text{——} \quad ** $$

On the other hand $\displaystyle \inf_{f \in A^0} I(f) \leq \inf_{\|f\|_p > 1} I(f)$

(Since $A^0 \subset \{\|f\|_p > 1\}$)

$$= \frac{1}{2} \inf_{c > (1+p)^{1/p}} c^2$$

$$= \frac{1}{2} \inf_{c \geq (1+p)^{1/p}} c^2 \quad \text{(monotonicity)}$$

$$= \inf_{f \in A} I(f) \quad \text{by } ** .$$

But, $A^0 \subset A \Rightarrow \displaystyle \inf_{f \in A^0} I(f) \geq \inf_{f \in A} I(f)$.

So, $\displaystyle \inf_{f \in A^0} I(f) = \inf_{f \in A} I(f)$. QED

This basically showed that the infimum is achieved on the boundary.

Alternatively, one can also argue as follows to show that the infimum is achieved on the boundary (and avoid the calculation with functional derivatives).

Assume that $\exists$ minimizing function $f_{min} \in A \setminus \partial A$.

Then $\| f_{min} \| > 1$

Then $\exists\, c > 1$, $\tilde{f} \in \partial A$ (ie $\|\tilde{f}\|_p = 1$) s.t. $f_{min} = c\tilde{f}$.

But $I(f_{min}) = c^2 I(\tilde{f})$.

Hence, $f_{min} \in \partial A$.

Now, get $(f_n)_{n\in\mathbb{N}}$ s.t. $f_n \in A$ s.t. $\|f_n - f\|_\infty \to 0$ and conclude $I(f_n) \to I(f)$.

ie $\|f_n - f\|_p \to 0$

This will show $\inf\limits_{f\in A^\circ} I(f) = \inf\limits_{f\in A} I(f)$.