Solution Sheet 1

Aufgabe 1:

(i) (a) $x \xmapsto{f} \mathbb{1}_{(0,1)}(x)$ is lower semicontinuous.

It is continuous everywhere except for 0 & 1.

$\liminf_{x_n \to 0} f(x_n) = 0 = f(0) < \limsup_{x_n \to 0} f(x_n) = 1$. Same reasoning works at $x = 1$

(b) $x \mapsto \mathbb{1}_{[0,1]}(x)$ is upper semicontinuous.

Same reasoning as above.

(c) $f(x) = \lfloor x \rfloor$ is upper semicontinuous, by definition:

$\forall x \in \mathbb{R} \quad \limsup_{x_n \to x} f(x_n) \leq f(x)$

(d) $f(x) = \lceil x \rceil$ is lower semicontinuous, same reasoning.

(ii) The function $g(x) = \begin{cases} -1 & x < 0 \\ -2 & x = 0 \\ -1/2 & x > 0 \end{cases}$ is lower-semicont.,

is continuous everywhere except 0, where it is neither left, nor right continuous.

(since $\liminf_{x_n \to 0} g(x_n) = -2 = g(0)$) though it is neither left, nor right continuous.

~~Aufgabe 2~~:

(ii) To show that: f l.s.c $\Leftrightarrow$ $\{f \leq \alpha\}$ is closed $\forall \alpha \in \mathbb{R}$.

($\Rightarrow$) Suppose $f(x_n) \leq \alpha$ $\forall n$ & $x_n \to x$. Then $f(x) \overset{(\text{l.s.c})}{\leq} \liminf_{x_n \to x} f(x_n) \leq \underbrace{\overline{\lim}_{x_n \to x} f(x_n)}_{\leq \alpha} \leq \alpha$.

$\Rightarrow \{f \leq \alpha\}$ closed.

($\Leftarrow$) Suppose $\forall \alpha \in \mathbb{R}$, $\{f \leq \alpha\}$ closed $\Leftrightarrow$ $\{f > \alpha\}$ open.

Fix $x_0 \in \mathbb{R}$ & Take $\alpha = f(x_0) - \varepsilon$ for some $\varepsilon > 0$. Clearly $x_0 \in \{f > f(x_0) - \varepsilon\}$ $\forall \varepsilon > 0$.

Now $\{f > \alpha\}$ open, for $\alpha = f(x_0) - \varepsilon$

$\Rightarrow \exists \delta > 0$ s.t. $\forall |y - x_0| < \delta$, $f(y) > f(x_0) - \varepsilon$.

Get a seq. $(x_n)_{n \in \mathbb{N}}$ s.t. $x_n \to x_0$ $\Rightarrow$ $|x_n - x_0| < \delta$ $\forall n > N$, for some $N \in \mathbb{N}$.

$\Rightarrow f(x_n) > f(x_0) - \varepsilon$ $\forall n > N$ $\Rightarrow \underline{\lim}_{n} f(x_n) \geq f(x_0) - \varepsilon$. Send $\varepsilon \downarrow 0$ to find the claim.

ii) $\{f_i\}_{i\in I}$ a lower semicont. family.

$f(x) = \sup_{i\in I} f_i(x)$, by definition.

Note, $f_i(x_n) \le \sup_{i\in I} f_i(x_n)$ $\forall n\in\mathbb{N}$ & $\forall i\in I$

$$\Rightarrow \lim_{x_n\to x_0} f_i(x_n) \le \lim_{x_n\to x_0} \sup_{i\in I} f_i(x_n)$$

But $\lim_{x_n\to x_0} f_i(x_n) \geq f(x_0)$ $\forall i$, f_i being all l.s.c

$$\Rightarrow f(x_0) \le \lim_{x_n\to x_0} \underbrace{\sup_{i\in I} f_i(x_n)}_{=: f(x_n)}$$

$\Rightarrow f$ l.s.c.

[True also for continuous functions: ie supremum of a family of continuous functions is lower semicontinuous (& need not be continuous!)]

iii) $K \subset \mathbb{R}$ compact. $f: K \to (-\infty, \infty]$ l.s.c.

$K\subset\mathbb{R}$ compact $\Rightarrow K = [a,b]$ for $a,b\in\mathbb{R}$.

~~Step~~ First, we claim that f is bounded below on $[a,b]$.
Suppose not. Then, $\forall n\in\mathbb{N}$, $\exists x_n \in [a,b]$ s.t $f(x_n) < -n$.
Since $(x_n)_{n\geq 1}$ lie in a bounded interval, get a subseq $(x_{nk})_{k\geq 1} \subset [a,b]$
s.t $x_{nk} \to x \in [a,b]$ ($x\in[a,b]$ as this is a closed interval).
Now, $\liminf_k f(x_{nk}) \geq f(x)$ by lower semicont.
But by assumption, $f(x_{nk}) < -n_k < -k \Rightarrow f(x) = -\infty$, contradiction.
Hence f is bounded below, so denote by $m = \inf_{[a,b]} f$.
Claim that $\exists x_0 \in [a,b]$ s.t. $f(x_0) = m$. Suppose not. ie $\forall x\in[a,b]$ $f(x) > m$. Again, $\forall n\in\mathbb{N}$, get $(x_n)_{n\geq 1} \subset [a,b]$ s.t. $f(x_n) \le m - 1/n$ (m is the inf)
But $f(x_n) > m \Rightarrow$ ~~&~~ $\lim f(x_n) = m$.
Get a subseq. $(x_{nk}) \subset [a,b]$ s.t. $x_{nk} \to x_0 \in [a,b]$. Then,
$m = \lim f(x_n) \overset{(l.s.c)}{\geq} f(x_0)$ contradiction.

Exercise 4 :

Take $(X_n)_n$ iid $\mathbb{P}(X_1 = 1) = 1/2 = \mathbb{P}(X_1 = -1)$

~~There~~ Let μ_n be the distribution of $\frac{S_n}{n}$.

Then $$\mu_n(\{0\}) = \begin{cases} 0 & n \text{ odd} \\ \binom{n}{n/2}\frac{1}{2^n} & n \text{ even} \end{cases}$$

Then $$\liminf_{n\to\infty} \frac{1}{n} \log \mu_n(\{0\}) = -\infty$$

$$\limsup_{n\to\infty} \frac{1}{n} \log \mu_n(\{0\}) = \lim_{n\to\infty} \frac{1}{n} \log\left\{\frac{n!}{\left(\frac{n}{2}\right)!\left(\frac{n}{2}\right)!}\right\} - \log 2$$

$$\frac{n!}{\left(\frac{n}{2}\right)!\left(\frac{n}{2}\right)!} \simeq \frac{n^n e^{-n}\sqrt{2\pi n}}{n^{n/2}\, n^{n/2}\, 2^{-n/2}\, 2^{-n/2}\, e^{-n/2}\, e^{-n/2}\, (2\pi n)}$$

$$= \frac{1}{2^{-n}\sqrt{2\pi}\sqrt{n}}$$

$$\Rightarrow \limsup \frac{1}{n} \log \mu_n(\{0\}) \neq -\infty.$$

Exercise 3:

$X \sim \mathcal{N}(0,1)$

$$\mathbb{P}(X > a) = \frac{1}{\sqrt{2\pi}} \int_a^{\infty} e^{-x^2/2}\, dx \leq \frac{1}{\sqrt{2\pi}} \int_a^{\infty} \frac{x}{a} e^{-x^2/2}\, dy = \frac{1}{\sqrt{2\pi}} \frac{1}{a} e^{-a^2/2}$$

Let $f(x) \overset{(\text{def})}{=} x e^{-x^2/2} - (x^2+1) \int_x^{\infty} e^{-y^2/2}\, dy$

Then $f(0) < 0$ & $\lim_{x \to \infty} f(x) = 0$

$$f'(x) = (1 - x^2 + x^2 + 1) e^{-x^2/2} - 2x \int_x^{\infty} e^{-u^2/2}\, du$$

$$= -2x \left(\int_x^{+\infty} e^{-y^2/2}\, dy - \frac{1}{x} e^{-x^2/2} \right)$$

$\Rightarrow f'(x) \geq 0 \quad \forall x \geq 0$ by the bound $\int_a^{\infty} e^{-x^2/2} dx \leq \frac{1}{a} e^{-a^2/2}$.

$\Rightarrow f(x) \leq 0$. $\boxed{}$