Solutions for Exercise sheet 2

Properties of Brownian Motion

Solution for Exercise 2.1

- Let us check the three Brownian motion properties for $X_t = \frac{1}{\sqrt{a}} B_{at}$, $a > 0$, $t \geq 0$:
  - (i) $X_0 = 0$ almost surely since $B_0 = 0$ almost surely.
  - (ii) Take $0 = t_0 < t_1 < ... < t_n$. The $(X_{t_i} - X_{t_{i-1}})_{i=1,\dots,n}$ are mutually independent since they are equal to $(\frac{1}{\sqrt{a}} B_{at_i} - \frac{1}{\sqrt{a}} B_{at_{i-1}})_{i=1,\dots,n}$ that are also independent by the definition of Brownian motion. Furthermore note that $X_i - X_{i-1} = \frac{1}{\sqrt{a}} B_{at_i} - \frac{1}{\sqrt{a}} B_{at_{i-1}} \sim \mathcal{N}(0, \left(\frac{1}{\sqrt{a}}\right)^2 a(t_i - t_{i-1}))$, so that we have the right distribution for the increments for $i = 1, ..., n$.
  - (iii) For $\mathcal{P}$-a.e. $\omega \in \Omega$ we have that the map $t \mapsto X_t(\omega) = \frac{1}{a} B_{\sqrt{a}t}(\omega)$ is continuous since it is a combination of continuous functions.
- And now we do the same for $Y_t = B_{a+t} - B_a$, $a > 0$.
  - (i) $Y_0 = B_{a+0} - B_a = 0$ (in particular, almost surely).
  - (ii) Take $0 = t_0 < t_1 < ... < t_n$. The $Y_{t_i} - Y_{t_{i-1}} = B_{t_i+a} - B_a - B_{t_{i-1}+a} + B_a = B_{t_i+a} - B_{t_{i-1}+a}$, so that independence of the increments and the fact that they are properly distributed follow.
  - (iii) For $\mathcal{P}$-a.e. $\omega \in \Omega$ we have that the map $t \mapsto Y_t(\omega) = B_{t+a} - B_a$ is continuous since it is a combination of continuous functions.

Solution for Exercise 2.2

(i) Write $X := B_{\frac{1}{3}} - 3B_4 + 5s = -3(B_4 - B_{\frac{1}{3}}) - 2B_{\frac{1}{3}} + 5s$. This is nothing but the sum of two independent Gaussians and a constant, therefore still a Gaussian (see, e.g., Homework Sheet 1). Its expectation is clearly $\mathsf{E}[X] = 5s$, while its variance is given by

$$\begin{aligned}\mathrm{Var}(X) &\overset{\text{(indep.)}}{=} \mathrm{Var}(-3(B_4 - B_{\frac{1}{3}})) + \mathrm{Var}(-2B_{\frac{1}{3}}) \\ &= 9\mathrm{Var}(\mathcal{N}(0, 4 - \tfrac{1}{3})) + 4\mathrm{Var}(\mathcal{N}(0, \tfrac{1}{3})) = \tfrac{103}{3}.\end{aligned}$$

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(ii) First note that, conditioned on B_2, B_4 becomes independent of B_1. Hence,

$$\begin{aligned}
\mathsf{E}[B_4^2|B_1,\, B_2] &= \mathsf{E}[B_4^2|B_2] \\
&= \mathsf{E}[(B_4 - B_2)^2 + B_2^2 + 2B_2(B_4 - B_2)|B_2] \\
&= \mathsf{E}[(B_4 - B_2)^2|B_2] + B_2^2 + 2B_2\mathsf{E}[B_4 - B_2|B_2] \\
&= 2 + B_2^2,
\end{aligned}$$

where we have used the fact that $(B_4 - B_2)$ conditioned on B_2 is distributed as a normal random variable $\mathcal{N}(0, 2)$ and that $\mathsf{E}[B_4|B_2] = B_2$.

If you still have some doubt, this is a good point for brushing up on the properties of conditional expectation!

Solution for Exercise 2.3

We call the process $(X_t)_{t\in[0,1]}$ the *Brownian Bridge* (draw a picture to understand why).

(i) Choose times $0 \le t_1 < t_2 < \ldots < t_n \le 1$. The strategy is similar to the one seen in the exercise class for proving that Brownian Motion is a Gaussian process: We find a linear transformation that transforms a known multivariate Gaussian into the desired vector.

$$\begin{pmatrix} X_{t_1} \\ X_{t_2} \\ \vdots \\ X_{t_n} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \ldots & 0 & -t_1 \\ 0 & 1 & \ldots & 0 & -t_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & -t_n \end{pmatrix} \begin{pmatrix} B_{t_1} \\ B_{t_2} \\ \vdots \\ B_{t_n} \\ B_1 \end{pmatrix}.$$

Since $(B_{t_1}, \ldots, B_{t_n}, B_1)$ is a multivariate Gaussian random vector, by Exercise 1.1 we know that $(X_{t_1}, \ldots, X_{t_n})$ is a multivariate Gaussian random vector, too. Therefore $(X_t)_{t\in[0,1]}$ is a Gaussian process.

(ii) For $0 \le s, t \le 1$ we have

$$\begin{aligned}
\mathbb{C}\text{ov}(X_s, X_t) &= \mathbb{C}\text{ov}(B_s - sB_1, B_t - tB_1) \\
&= \mathbb{C}\text{ov}(B_s, B_t) - s\mathbb{C}\text{ov}(B_1, B_t) - t\mathbb{C}\text{ov}(B_s, B_1) + st\text{Var}\,(B_1) \\
&= s \wedge t - st - ts + st \\
&= s \wedge t - st \ge 0.
\end{aligned}$$

In particular we see that $\text{Var}\,(X_t) = t - t^2$.

(iii) In order to show that the process $(X_t)_{t\in[0,1]}$ does not have independent increments, we can calculate the covariance of two increments. Let $0 < s < t$ and use part ii) to calculate

$$\begin{aligned}
\mathbb{C}\text{ov}(X_t - X_s, X_s) &= \mathbb{C}\text{ov}(X_t, X_s) - \mathbb{C}\text{ov}(X_s, X_s) \\
&= s \wedge t - st - (s - s^2) \\
&= s(s - t) \neq 0.
\end{aligned}$$

Thus the increments cannot be independent.

Solution for Exercise 2.4

Here is one of the possible solutions: Fix any $t > 0$. To compute the distribution function $F_{R_t}(x) := P(R_t \leq x)$ for any $x \geq 0$, we observe that

$$F_{R_t}(x) := P(R_t^2 \leq x^2) = P((B_t^1)^2 + (B_t^2)^2 \leq x^2) = P((B_t^1, B_t^2) \in D(x)),$$

where $D(x)$ is the disk of radius x centered at the origin. Since B^1 and B^2 are independent, their joint density is

$$f_{(B_t^1,B_t^2)}(x_1, x_2) = \varphi_t(x_1)\varphi_t(x_2),$$

where φ_t is the density of a $\mathcal{N}(0, t)$-distributed random variable. Therefore,

$$f_{(B_t^1,B_t^2)}(x_1, x_2) = \frac{1}{\sqrt{2\pi t}} \mathrm{e}^{-\frac{x_1^2}{2t}} \cdot \frac{1}{\sqrt{2\pi t}} \mathrm{e}^{-\frac{x_2^2}{2t}}.$$

Thus, we obtain that

$$F_{R_t}(x) = \int\int_{x_1^2+x_2^2 \leq x^2} \frac{1}{2\pi t} \mathrm{e}^{-\frac{x_1^2+x_2^2}{2t}} \,\mathrm{d}x_1 \mathrm{d}x_2.$$

Passing to polar coordinates yields

$$F_{R_t}(x) = \int_0^{2\pi} \int_0^x \frac{r}{2\pi t} \mathrm{e}^{-\frac{r^2}{2t}} \,\mathrm{d}r \mathrm{d}\theta = \int_0^x \frac{r}{t} \mathrm{e}^{-\frac{r^2}{2t}} \,\mathrm{d}r.$$

By differentiating with respect to x, we see that

$$f_{R_t}(x) = \frac{\mathrm{d}}{\mathrm{d}x} F_{R_t}(x) = \frac{x}{t} \,\mathrm{e}^{-\frac{x^2}{2t}}, \qquad x \geq 0.$$

Exercise 2.8 (Bonus!)

Somebody asked me to publish also the solutions for Exercise 2.8, since we did not manage to finish to do it in class (try to do the exercise by yourself first!).

We first remark that every process is adapted to its associated (also natural or generated) filtration, i.e. $(B_t)_{t\geq 0}$ is adapted to $(\mathcal{F}_t)_{t\geq 0}$. Furthermore, we will repeatedly use that for all $0 \leq s \leq t$ the increment $B_t - B_s$ is independent of $\mathcal{F}_s$ which will be properly shown in Exercise 3.1.

(i) The process $(B_t)_{t\geq 0}$ is adapted to $(\mathcal{F}_t)_{t\geq 0}$. Using Jensen's inequality we get for all $t \geq 0$

$$\mathsf{E}[|B_t|] = \mathsf{E}[\sqrt{B_t^2}] \leq \sqrt{\mathsf{E}[B_t^2]} = \sqrt{t} < \infty$$

and for all $0 \leq s \leq t$

$$\mathsf{E}[B_t | F_s] = \mathsf{E}[B_t - B_s | \mathcal{F}_s] + B_s = B_s.$$

(ii) $(B_t^2 - t)_{t\geq 0}$ is adapted to $(\mathcal{F}_t)_{t\geq 0}$ since it is the composition of measurable functions. Also for all $t \geq 0$

$$\mathsf{E}[|B_t^2 - t|] \leq \mathsf{E}[B_t^2] + t = 2t < \infty,$$

and for all $0 \leq s \leq t$

$$\mathsf{E}[B_t^2 - t|\mathcal{F}_s] = \mathsf{E}[(B_t - B_s)^2 - 2(B_t - B_s)B_s + B_s^2|\mathcal{F}_s] - t = (t-s) + B_s^2 - t = B_s^2 - s.$$

(iii) $(M_t)_{t\geq 0}$ is adapted to $(\mathcal{F}_t)_{t\geq 0}$ since it is the composition of measurable functions. We use our result from Exercise 2.7, i.e.

$$\mathsf{E}[\exp(\sigma B_s)] = \exp\left(\frac{1}{2}\sigma^2 s\right).$$

For all $t \geq 0$

$$\mathsf{E}[|\exp(\alpha B_t - \alpha^2 t/2)|] = \exp(-\alpha^2 t/2)\mathsf{E}[\exp(\alpha B_t)] = \exp(\alpha^2(t-t)/2) = 1 < \infty,$$

and for all $0 \leq s \leq t$

$$\begin{aligned}\mathsf{E}[\exp(\alpha B_t - \alpha^2 t/2)|\mathcal{F}_s] &= \exp(\alpha^2 t/2)\mathsf{E}[\exp(\alpha(B_t - B_s) + \alpha B_s)|\mathcal{F}_s] \\ &= \exp(\alpha^2 t/2)\exp(\alpha^2(t-s)/2)\exp(\alpha B_s) \\ &= \exp(\alpha B_s - \alpha^2 a/2).\end{aligned}$$