Solutions for Exercise sheet 3
Properties of Brownian Motion

For the first exercise we can use the following theorem from Probability Theory.

Theorem 1 *Assume that G_i, $i \in I$, is an independent collection of systems of sets and G_i is stable under countable intersections ($\cap$-stable) for every $i \in I$. Then*

(i) The σ-fields $\sigma(G_i)$, $i \in I$ are independent and

(ii) for a partition J_k, $k \in K$ of I in pairwise disjoint subsets, the σ-fields

$$\sigma\left(\bigcup_{i \in J_k} G_i\right), \quad k \in K$$

are independent.

Solution for Exercise 3.1

(i) We first show $\mathcal{F}_s \subseteq \sigma(\mathcal{M}_s)$. Let us consider some $0 \le t \le s$. We have $\{B_t \in A\} \in \mathcal{M}_s$ for all $A \in \mathcal{B}(\mathbb{R})$. Thus B_t is $\sigma(\mathcal{M}_s)$-measurable. Since this is true for all $0 \le t \le s$ we conclude that

$$\mathcal{F}_s = \sigma(B_t : 0 \le t \le s) \subseteq \sigma(\mathcal{M}_s).$$

To show the other direction, $\sigma(\mathcal{M}_s) \subseteq \mathcal{F}_s$, we consider an arbitrary set $M \in \mathcal{M}_s$, $0 = t_0 < t_1 < \ldots < t_n$,

$$\begin{aligned} M &= \{B_{t_1} \in A_1, B_{t_2} - B_{t_1} \in A_2, \ldots, B_{t_n} - B_{t_{n-1}} \in A_n\} \\ &= \bigcap_{i=1}^{n} \{B_{t_i} - B_{t_{i-1}} \in A_i\} \\ &\subseteq \bigcap_{i=1}^{n} \sigma(B_{t_{i-1}}, B_{t_i}) \\ &\subseteq \sigma(B_{t_1}, B_{t_2}, \ldots, B_{t_n}) \subseteq \mathcal{F}_s, \end{aligned}$$

since M is a countable intersection of measurable sets.

(ii) We define two collections of sets

$$\begin{aligned} G_1 := \{&B_{t_1} \in A_1, \ldots, B_{t_n} \in A_n : A_1, \ldots, A_n \in \mathcal{B}(\mathbb{R}), \\ &n \in \mathbb{N}, 0 \le t_1 < t_2 < \ldots < t_n \le s\} \quad \text{and} \\ G_2 := \{&B_{s+t_1} - B_s \in A_1, \ldots, B_{s+t_n} - B_s \in A_n : A_1, \ldots, A_n \in \mathcal{B}(\mathbb{R}), \\ &n \in \mathbb{N}, 0 \le t_1 < t_2 < \ldots < t_n\} \end{aligned}$$

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and observe that they are ∩-stable, since

$$\{B_{t_1} \in A_1\} \cap \{B_{t_2} \in A_2\} = \{B_{t_1} \in A_1, B_{t_2} \in A_2\}$$
$$\{B_{t_1} \in A_1\} \cap \{B_{t_1} \in A_1\} = \{B_{t_1} \in A_1 \cap A_2\}$$
$$\vdots$$

Now we need to show that for any two sets $M_1 \in G_1$ and $M_2 \in G_2$ are independent. Thus we fix $n, j \in \mathbb{N}$, $s > 0$, $0 \leq t_1 < t_2 < \ldots < t_n \leq s$, $0 \leq v_1 < v_2 < \ldots < v_j$, $A_1, \ldots, A_n, C_1, \ldots, C_n \in \mathcal{B}(\mathbb{R})$ and set

$$M_1 = \{B_{t_1} \in A_1, \ldots, B_{t_n} \in A_n\}$$
$$M_2 = \{B_{s+v_1} \in C_1, \ldots, B_{s+v_n} \in C_n\}$$

Thus

$$M_1 \in \sigma(B_{t_1}, B_{t_2}, \ldots, B_{t_n}) = \sigma(B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}})$$
$$M_2 \in \sigma(B_{s+v_1}, B_{s+v_2}, \ldots, B_{s+v_n}) = \sigma(B_{s+v_1}, B_{s+v_2} - B_{s+v_1}, \ldots, B_{s+v_n} - B_{s+v_{n-1}})$$

By part ii) of the above theorem M_1 and M_2 are independent. Since we have chosen M_1 and M_2 arbitrary, also G_1 and G_2 are independent. By part i) of the theorem we know that $\sigma(G_1)$ and $\sigma(G_2)$ are independent. Finally we observe, that by part i) of this exercise $\sigma(G_1) = \sigma(M_s) = \mathcal{F}_s$ and $\sigma(G_2) = \sigma((B_{t+s} - B_s)_{t \geq 0})$.

Solution for Exercise 3.2

(i) We show that $\mathrm{I}_{\{\overline{\lim}_{t\to\infty} X_t \geq K\}}(\omega) \geq \overline{\lim}_{t\to\infty} \mathrm{I}_{\{X_t \geq K\}}(\omega)$. In fact,

$$\mathrm{I}_{\{\overline{\lim}_{t\to\infty} X_t \geq K\}}(\omega) = \begin{cases} 1 & \text{if, for all } \varepsilon > 0\text{, there exists } t \text{ arbitrarely large} \\ & \text{such that } X_t(\omega) \geq K - \varepsilon, \\ 0 & \text{otherwise.} \end{cases}$$

while

$$\overline{\lim_{t\to\infty}} \mathrm{I}_{\{X_t \geq K\}}(\omega) = \begin{cases} 1 & \text{if there exists } t \text{ arbitrarely large such that } X_t(\omega) \geq K, \\ 0 & \text{otherwise.} \end{cases}$$

Observe for example that for ω such that $X_t(\omega) = K - \frac{1}{t}$, we have that the first indicator function is equal to 1, while the second is equal to 0.

(ii) We first observe that

$$\overline{\lim_{t\to\infty}} \frac{B_t}{\sqrt{t}} \geq \overline{\lim_{t\to\infty}} \frac{B_{\lfloor t \rfloor}}{\sqrt{\lfloor t \rfloor}} = \overline{\lim_{n\to\infty,\, n\in\mathbb{N}}} \frac{B_n}{\sqrt{n}}.$$

Since

$$B_n = \sum_{k=1}^{n} Y_i$$

is the sum of i.i.d. random variables Y_i with law $\mathcal{N}(0,1)$, we know that

$$\lim_{n\to\infty} \frac{B_n}{\sqrt{n}} \sim \mathcal{N}(0,1)$$

by the Central Limit Theorem. We set $X \sim \mathcal{N}(0,1)$, fix $K > 0$ ans some $a > 0$ and estimate

$$\begin{aligned}
\mathsf{P}\left(\overline{\lim_{t\to\infty}} \frac{B_t}{\sqrt{t}} \geq K\right) &\geq \mathsf{P}\left(\overline{\lim_{n\to\infty}} \frac{B_n}{\sqrt{n}} \geq K\right) \\
&\geq \overline{\lim_{t\to\infty}}\, \mathsf{P}\left(\frac{B_n}{\sqrt{n}} \geq K + a\right) \\
&= \mathsf{P}(X \geq K + a) > 0.
\end{aligned}$$

For the second step we use the fact that

$$\mathrm{I}_{\left\{\overline{\lim}_{n\to\infty} \frac{B_n}{\sqrt{n}} \geq K\right\}} \geq \overline{\lim_{n\to\infty}}\, \mathrm{I}_{\left\{\frac{B_n}{\sqrt{n}} \geq K+a\right\}}$$

and the Lemma of Fatou (note that we have to use the $a > 0$ for what we have seen in part (i) of the exercise!).

(iii) Finally, we want to apply Kolomogorov's 0-1 law. Note that the event

$$\left\{\overline{\lim_{n\to\infty}} \frac{\sum_{k=1}^{M} Y_i}{\sqrt{n}} \geq K\right\}$$

does not depend on the first M variables Y_i, for any fixed $M \in \mathbb{N}$. In fact, we can always split the sum as $\frac{1}{\sqrt{n}}\sum_{k=1}^{n} Y_i = \frac{1}{\sqrt{n}}\sum_{k=1}^{M} Y_i + \frac{1}{\sqrt{n}}\sum_{k=M+1}^{n} Y_i$ and the first part vanishes as $n \to \infty$, giving no contribution to the $\overline{\lim}$. Hence, $\left\{\overline{\lim}_{n\to\infty} \frac{B_n}{\sqrt{n}} \geq K\right\}$ is in the tail σ-algebra $\tau_\infty = \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} \sigma(Y_i)$ generated by the Y_i. We can conclude by Kolomogorov's 0-1 law that for all $K > 0$

$$\mathsf{P}\left(\overline{\lim_{t\to\infty}} \frac{B_t}{\sqrt{t}} \geq K\right) = 1$$

and therefore, by continuity of the measure,

$$\mathsf{P}\left(\overline{\lim_{t\to\infty}} \frac{B_t}{\sqrt{t}} = \infty\right) = 1.$$

Note that, by symmetry of Brownian motion, we also have

$$\mathsf{P}\left(\underline{\lim_{t\to\infty}} \frac{B_t}{\sqrt{t}} = -\infty\right) = 1;$$

this, together with the continuity property, implies that the Brownian Motion hits 0 infinitely many times, for arbitrarely large t's.

Solution for Exercise 3.3

The event E has probability 0. We will procede as follows: Prove that $E \in \mathcal{F}^+(0)$, use Blumenthal to infer that it has probability either 0 or 1 and then show why it must be 0.
First of all rewrite

$$
\begin{aligned}
E &= \{\exists \varepsilon > 0 : \forall 0 < h < \varepsilon, \max_{t\in[0,h]} B_t > |\min_{t\in[0,h]} B_t|\} \\
&= \{\exists N \in \mathbb{N} : \forall 0 < h < \tfrac{1}{N}, \max_{t\in[0,h]} B_t > |\min_{t\in[0,h]} B_t|\} \\
&= \bigcup_{N\in\mathbb{N}} \{\forall 0 < h < \tfrac{1}{N}, \max_{t\in[0,h]} B_t > |\min_{t\in[0,h]} B_t|\} \\
&= \bigcup_{N\in\mathbb{N}} \bigcap_{h\in\left[0,\frac{1}{N}\right]} \{\max_{t\in[0,h]} B_t > |\min_{t\in[0,h]} B_t|\}. \qquad (1)
\end{aligned}
$$

The event $\{\max_{t\in[0,h]} B_t > |\min_{t\in[0,h]} B_t|\}$ clearly depends only on the history of the Brownian Motion up to time h, and hence belongs to $\mathcal{F}^0(h)$. Hence, for every $N \in \mathbb{N}$, the intersection in equation (1) belongs to $\mathcal{F}^+(0)$. The union of countably many sets of a σ-algebra still belongs to the same σ-algebra, so that overall $E \in \mathcal{F}^+(0)$.
By Blumenthal 0-1 law we know that the event E must have either probablity 0 or 1.
Consider now the event

$$F := \{\exists \varepsilon > 0 : \forall 0 < h < \varepsilon, \max_{t\in[0,h]} B_t < |\min_{t\in[0,h]} B_t|\}.$$

F must have the same probability as E because of the symmetry of Brownian Motion. On the other hand F and E are clearly disjoint, so they cannot have both probability 1. The only possibility left is that they both have probability 0.

Note: The phylosophical message of this exercise is (more or less) the following: It does not matter how hard you search, you will not find a notion of 'starting to go up' such that -with positive probability- a Brownian Motion 'starts to go up'. Brownian Motion goes up and down from the beginning infinitely many times. Brownian Motion is crazy.

Solution for Exercise 3.4

In order to find the right f, we start computing the expectation of B_t^3 given the history of the Brownian Motion up to time $s < t$:. Let $W := B_t - B_s$. W is a normal random variable with mean 0 and variance $t - s$. Note that by Exercise 3.1 W is independent of $\mathcal{F}_s$.

$$
\begin{aligned}
\mathbb{E}[B_t^3|\mathcal{F}_s] &= \mathbb{E}[(B_s + W)^3|\mathcal{F}_s] \\
&= \mathbb{E}[B_s^3 + 3B_s^2 W + 3B_s W^2 + W^3|\mathcal{F}_s] \\
&= B_s^3 + 3B_s^2\mathbb{E}[W] + 3B_s\mathbb{E}[W^2] + \mathbb{E}[W^3],
\end{aligned}
$$

where we have used the linearity of the expectation, the fact that B_s is $\mathcal{F}_s$-measurable (and therefore "drops outöf the conditional expectation) and that W is independent of $\mathcal{F}_s$. The expectations of W and W^3 are equal to 0 (see Exercise 1.2), while the expectation of W^2 is $t-s$. Therefore

$$\mathbb{E}[B_t^3|\mathcal{F}_s] = B_s^3 + 3B_s(t-s).$$

It follows that if we choose $f(t) = -3t$ we get

$$\mathbb{E}[X_t|\mathcal{F}_s] = \mathbb{E}[B_t^3 + f(t)B_t|\mathcal{F}_s] = B_s^3 + 3B_s(t-s) + f(t)B_s = B_s^3 - 3B_s = X_s,$$

so that $(X_t)_{t\geq 0}$ has the martingale property.
We are left to check that (X_t) is adapted (which is clear, since X_t is the combination of the measurable functions B_t^3, $3t$ and B_t) and integrable. This last property is easy to check:

$$\mathbb{E}[|X_t|] = \mathbb{E}[|B_t^3 - 3tB_t|] \leq \mathbb{E}[|B_t|^3] + 3t\mathbb{E}[|B_t|].$$

We have already seen in calss how to bound $\mathbb{E}[|B_t|]$, e.g. with Jensen. Finally we have

$$\mathbb{E}[|B_t|^3] = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} |x|^3 \mathrm{e}^{-\frac{x^2}{2t}} \, \mathrm{d}x,$$

which is also a finite quantity for all $t > 0$ (if you are not convinced you can just make the easy calculation (note that you are integrating an even function on the whole of $\mathbb{R}$!)).