TU München
Zentrum Mathematik
Lehrstuhl für Wahrscheinlichkeitstheorie

Stochastic Analysis
Winter Term 2014/15
Berger/Salvi

Exercise sheet 1
Multivariate Gaussian Random Variables

Information:

- **Registration:** You will be able to sign up for the exercise classes via TUMonline starting from the 10th of October 2014, at 16h00. There will be 3 Groups:
  Group 1 takes place on even weeks, from 12h30 to 14h00. Start: 16.10.2014.
  Group 2 takes place on odd weeks, from 08h30 to 10h00. Start: 23.10.2014.
  Group 3 takes place on odd weeks, from 12h30 to 14h00. Start: 23.10.2014.
  All the exercise classes will beheld in 2.01.10, Hochbrück-Tutorraum 1 (8101.02.110).
- **Website:** http://www-m14.ma.tum.de/en/teaching/ws14-15/stochastic-analysis/
- **Bonus system:** Please look up the details of the bonus system for the exams on our website.

Homework:
Please hand in the solutions of the homework in *groups of two or three people*. The solutions of Exercise 1.1 and 1.2 are due on October 16 (hand them in just after the lecture).

Definition
A random vector $\vec{X} := (X_1, X_2, \dots, X_n)^{tr}$ has a (multivariate) normal distribution iff

$$\langle (X_1, X_2, \dots, X_n), (a_1, a_2, \dots, a_n) \rangle = \sum_{i=1}^{n} a_i X_i$$

has a one-dimensional normal distribution for all $a_1, a_2, \dots, a_n \in \mathbb{R}$. We write $\vec{X} \sim \mathcal{N}(\vec{\mu}, \Sigma)$ where $\vec{\mu} := (\mathsf{E}[X_1], \mathsf{E}[X_2], \dots, \mathsf{E}[X_n])^{tr}$ is the mean vector of $\vec{X}$ and

$$\Sigma := \begin{pmatrix} \mathbb{C}\mathrm{ov}(X_1, X_1) & \mathbb{C}\mathrm{ov}(X_1, X_2) & \dots & \mathbb{C}\mathrm{ov}(X_1, X_n) \\ \mathbb{C}\mathrm{ov}(X_2, X_1) & \mathbb{C}\mathrm{ov}(X_2, X_2) & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \mathbb{C}\mathrm{ov}(X_n, X_1) & \dots & \dots & \mathbb{C}\mathrm{ov}(X_n, X_n) \end{pmatrix}$$

the covariance matrix of $\vec{X}$ (see also Exercise 1.5 (i)).

Exercise 1.1 (6 Points)

(i) Consider a random vector

$$\vec{X} := \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix}$$

with a $\mathcal{N}(\vec{\mu}, \Sigma)$ distribution, some matrix $A \in \mathbb{R}^{k \times n}$ and some vector $\vec{b} \in \mathbb{R}^k$. Determine the distribution of $A\vec{X} + \vec{b}$.

Hint: You may use that the characteristic function $\varphi : \mathbb{R}^n \to \mathbb{C}$ of a n-dimensional $\mathcal{N}(\vec{\mu}, \Sigma)$-distribution is given by

$$\varphi(\vec{t}\,) = \exp\left(i \cdot \vec{t}^{\,tr} \cdot \vec{\mu} - \tfrac{1}{2} \cdot \vec{t}^{\,tr} \cdot \Sigma \cdot \vec{t}\,\right).$$

The characteristic function of a random vector determines its distribution uniquely.

(ii) Consider a two-dimensional random vector (X_1, X_2) with common density

$$f(x_1, x_2) := \frac{\sqrt{7}}{2\pi} \exp\left(-2x_1^2 - x_2^2 - x_1 x_2\right).$$

Determine the marginal distributions of X_1 and X_2. Are X_1 and X_2 stochastically independent?

Hint: Recall the following fact about multivariate normal distributions: Consider a random vector $\vec{X} := (X_1, X_2, \ldots, X_n)^{tr}$ where $\vec{X} \sim \mathcal{N}(\vec{\mu}, \Sigma)$. If Σ is not singular, i.e. $\det(\Sigma) > 0$ holds, then the inverse Σ^{-1} exists and $\vec{X}$ has the density

$$f_{\vec{X}}(\vec{x}) = \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \cdot \frac{1}{\sqrt{\det(\Sigma)}} \cdot \exp\left(-\frac{1}{2}(\vec{x} - \vec{\mu})^{tr} \cdot \Sigma^{-1} \cdot (\vec{x} - \vec{\mu})\right)$$

with respect to the n-dimensional Lebesgue-measure.

Exercise 1.2 (4 Points)
Let $X \sim \mathcal{N}(0, \sigma^2)$. Compute $\mathsf{E}\left[X^k\right]$ for $k \in \mathbb{N}$.

Exercises for the tutorial:
Exercises 1.3 - 1.5 will be discussed in the tutorials on 16th (Group 1) and 23rd of October (Groups 2 and 3).

Exercise 1.3
Find two univariate Gaussian random variables X, Y such that

(i) X and Y are dependent and (X, Y) is bivariate Gaussian.

(ii) X and Y have a density, (X, Y) is bivariate Gaussian and has no density.

(iii) (X, Y) is not bivariate Gaussian.

Exercise 1.4
Consider two independent random variables $X \sim \mathcal{N}(0, \sigma_1^2)$ and $Y \sim \mathcal{N}(0, \sigma_2^2)$ for some $\sigma_1^2 > 0$ and $\sigma_2^2 > 0$. Compute the distribution of $X + Y$ and $X - Y$.

Exercise 1.5

(i) Show that any multivariate Gaussian random vector is uniquely determined by its mean and covariance matrix.

(ii) Show that any matrix is a covariance matrix iff it is positive semi-definite and symmetric.