Exercise sheet 2

Properties of Brownian Motion

Homework:
Please hand in the solutions of Exercises from 2.1 to 2.4 in the lecture on October 30th.

Exercise 2.1 (6 Points)
Let $(B_t)_{t\geq 0}$ be a Brownian motion and $a > 0$. Show that $(\frac{1}{\sqrt{a}}B_{at})_{t\geq 0}$ and $(B_{a+t} - B_a)_{t\geq 0}$ are Brownian motions.

Exercise 2.2 (4 Points)
Let $(B_t)_{t\geq 0}$ be a Brownian motion.

(i) Compute the distribution of $B_{\frac{1}{3}} - 3B_4 + 5s$ for some $s \in \mathbb{R}$.

(ii) Compute $\mathsf{E}\big[(B_4)^2\big|B_1, B_2\big]$.

Exercise 2.3 (6 Points)
Let $(B_t)_{t\geq 0}$ be Brownian motion and consider a process $(X_t)_{0\leq t\leq 1}$ where

$$X_t := B_t - tB_1 \quad \text{for } 0 \leq t \leq 1.$$

(i) Show that $(X_t)_{0\leq t\leq 1}$ is a Gaussian process

(ii) Compute the covariance $\mathbb{C}\mathrm{ov}(X_s, X_t)$ for $0 \leq s, t \leq 1$.

(iii) Show that $(X_t)_{0\leq t\leq 1}$ does not have independent increments.

Definition
Two stochastic processes $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ are called *independent* if for any choice of times $t_1, ..., t_n \geq 0$ and $s_1, ..., s_m \geq 0$, the vectors $(X_{t_1}, ..., X_{t_n})$ and $(Y_{s_1}, ..., Y_{s_m})$ are independent.

Exercise 2.4 (4 Points)
Let $(B^1_t)_{t\geq 0}$ and $(B^2_t)_{t\geq 0}$ be two independent Brownian motions. You can think of the trajectory $t \to (B^1_t(\omega),\, B^1_t(\omega))$ as the motion of a particle in the plane (we call it *planar Brownian Motion*). Let the stochastic process $(R_t)_{0\leq t\leq T}$ denote the distance between the Brownian particle $(B^1,\, B^2)$ and the origin, i.e., for each $t \in [0, T]$,

$$R_t := \sqrt{(B^1_t)^2 + (B^2_t)^2}.$$

For each fixed $t \in (0, T]$, find the density f_{R_t} of R_t.

Exercises for the tutorial:
Exercises 2.5 - 2.7 will be discussed in the tutorials on October the 30th (Group 1) and November the 6th (Groups 2 and 3).

Exercise 2.5
Let $(B_t)_{t\geq 0}$ be a Brownian motion and $\sigma \in \mathbb{R}$. Compute

$$\mathsf{E}\big[\exp\big(\sigma(B_t - B_s)\big)\big], \qquad t > s \geq 0.$$

Exercise 2.6
For $(B_t)_{t\geq 0}$ standard Brownian Motion, define the stochastic process

$$X_t := \begin{cases} tB_{1/t} & \text{for } t > 0 \\ 0 & \text{for } t = 0. \end{cases}$$

Show that $(X_t)_{t\geq 0}$ is also a Brownian Motion.

Exercise 2.7

(i) Let $\tau_1 := \inf\{t > 0 : \ B_t > 0\}$. Show that $P_0(\tau_1 = 0) = 1$.

(ii) Let $\tau_2 := \inf\{t > 0 : \ B_t = 0\}$. Show that $P_0(\tau_2 = 0) = 1$.

Exercise 2.8
Consider a Brownian motion $(B_t)_{t\geq 0}$ and the associated canonical filtration $(\mathcal{F}_t)_{t\geq 0}$, i.e.

$$\mathcal{F}_t := \sigma(B_s : \ 0 \leq s \leq t) \quad \text{for } t \geq 0.$$

Show that the following stochastic processes are martingales with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$:

(i) $(B_t)_{t\geq 0}$

(ii) $(B_t^2 - t)_{t\geq 0}$

(iii) Fix $\alpha \in \mathbb{R}$ and consider $(M_t)_{t\geq 0}$ where $M_t := \exp\big(\alpha B_t - \frac{1}{2}\alpha^2 t\big)$.